1 Problem 33

1.1 Problem 33, 8th edition

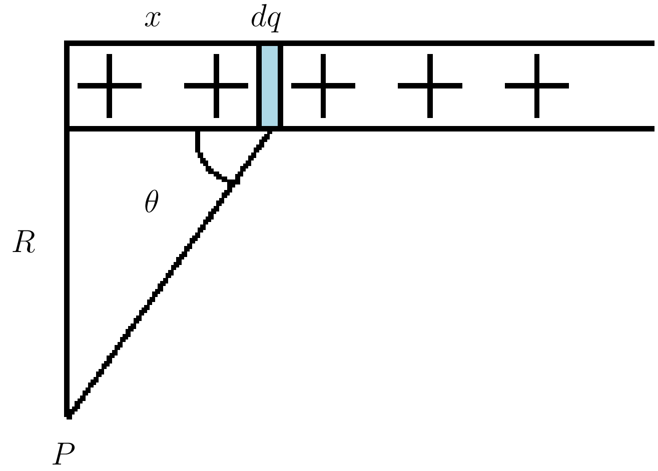

Figure 1: From book

- Charge density is λ

We want to show:

> The electric field at P makes an angle of $45°$ with the rod, independent of R.

BEFORE MATH

Things to note:

1. To show that the electric field is at $45°$ requires showing that the x and y components are equal, independent of R.
2. The total field is the integral of the infinitesimal field contributions.

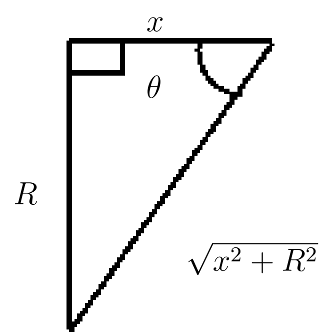

Figure 2: Resolving force into components

CALCULATIONS

To determine the field components, we have to resolve into components.

$$d_{E_P} = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{dq}{x^2+R^2}$$

$$dq = \lambda dx$$

So,

$$d_{E_P} = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{\lambda dx}{x^2+R^2}$$

$$d_{E_{P_x}} = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{\lambda dx}{x^2+R^2}\frac{x}{\sqrt{x^2+R^2}}$$

$$d_{E_{P_y}} = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{\lambda dx}{x^2+R^2}\frac{R}{\sqrt{x^2+R^2}}$$

To do the integration, it will probably be easier to integrate in θ rather than in x. **In cases like this, the variable substitution will make more algebra but easier integration. Without the substitution there will be less algebra but a tougher integration.**

VARIABLE SUBSTITUTION

$$\sin\theta = \frac{R}{\sqrt{x^2+R^2}}$$

$$\cos\theta = \frac{x}{\sqrt{x^2+R^2}}$$

When $x = 0$, $\theta = \frac{\pi}{2}$.
As $x \to \infty$, $\theta \to 0$.

$$\tan\theta = \frac{R}{x}$$

We can differentiate one of the above equations to relate $d\theta$ and dx.

$$\tan\theta = \frac{R}{x} \to \sec^2\theta \quad d\theta = -\frac{R}{x^2} \quad dx$$

or

$$\sec^2\theta \quad d\theta = -\left(1/R\right)\frac{R^2}{x^2} \quad dx = -\left(1/R\right)\tan^2\theta \quad dx$$

and so

$$dx = R \quad \sec^2\theta \quad d\theta \quad \frac{1}{\tan^2\theta} = R \quad \frac{1}{\cancel{\cos^2\theta}} \ \frac{\cancel{\cos^2\theta}}{\sin^2\theta} \quad d\theta = \frac{R}{\sin^2\theta} \quad d\theta$$

So we can rewrite our equations for $d_{E_{P_x}}$ and $d_{E_{P_y}}$ in terms of θ.

$$d_{E_{P_x}} = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{\lambda dx}{x^2+R^2}\frac{x}{\sqrt{x^2+R^2}} = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{\lambda\frac{R}{\sin^2\theta} \quad d\theta}{\frac{R^2}{\sin^2\theta}}\cos\theta$$

Thus

$$d_{E_{P_x}} = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{\lambda}{R}\cos\theta \quad d\theta$$

Note that the variable change has simplified this greatly.

Similarly,

$$d_{E_{P_y}} = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{\lambda dx}{x^2+R^2}\frac{R}{\sqrt{x^2+R^2}} = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{\lambda\frac{R}{\sin^2\theta} \quad d\theta}{\frac{R^2}{\sin^2\theta}}\sin\theta$$

Thus

$$d_{E_{P_y}} = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{\lambda}{R}\sin\theta \quad d\theta$$

$$E_{P_x} = \int_0^{\pi/2}\left(\frac{1}{4\pi\epsilon_0}\right)\frac{\lambda}{R}\cos\theta \quad d\theta = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{\lambda}{R}\int_0^{\pi/2}\cos\theta \quad d\theta$$

$$E_{P_x} = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{\lambda}{R}\int_0^{\pi/2}\cos\theta \quad d\theta = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{\lambda}{R}[\sin\theta]_0^{\pi/2}$$

so

$$E_{P_x} = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{\lambda}{R}[1-0] = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{\lambda}{R}$$

$$E_{P_y} = \int_0^{\pi/2}\left(\frac{1}{4\pi\epsilon_0}\right)\frac{\lambda}{R}\sin\theta \quad d\theta = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{\lambda}{R}\int_0^{\pi/2}\sin\theta \quad d\theta$$

$$E_{P_y} = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{\lambda}{R}\int_0^{\pi/2}\sin\theta \quad d\theta = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{\lambda}{R}[-\cos\theta]_0^{\pi/2}$$

so

$$E_{P_y} = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{\lambda}{R}[-0-(-1)] = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{\lambda}{R}$$

The two are equal, as required.