Electronics
Kirchhoff's Law Example

Terry Sturtevant

Wilfrid Laurier University

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Analyzing the behaviour of DC circuits involving *only* DC (ie. unchanging in time) voltage sources and **ohmic** devices can be done using **Kirchhoff's Laws**.

All of these calculations assume that the current through a device is ***proportional*** *to the voltage across the device.*

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For many devices the relationship may be far more complex, and so the equations relating voltages and currents in the circuit become non-linear and are much harder to solve.

These laws are

At any point in a circuit, the sum of the currents flowing toward the point is equal to the sum of the currents flowing away from the point; or, the algebraic sum of all the currents flowing toward a node is equal to zero.

$$\sum I = 0$$

In any closed circuit, the algebraic sum of all the voltages around the loop is equal to zero; or, the sum of all voltage sources is equal to the sum of all IR drops[1]*.* (In this case "voltage sources" can be positive or negative.)

$$\sum V = \sum IR$$

[1] The path around a loop is a series of voltage "steps", in which, at the end of the loop, you must return to the same voltage you started with. There are only two kinds of voltages in the circuit. Those due to **sources**, such as batteries, and those due to **sinks**, which are resistors. By Ohm's law, the voltage across a resistor is IR.

The way that these laws are applied to analyze a circuit involves choosing **nodes** in the circuit for the first law and using **loops** in the circuit for the second and producing equations from each node and loop.

After the equations are created, determining the currents in the circuit is accomplished by solving the equations simultaneously. In this case, the system we must solve is of the form

$$AX = B$$

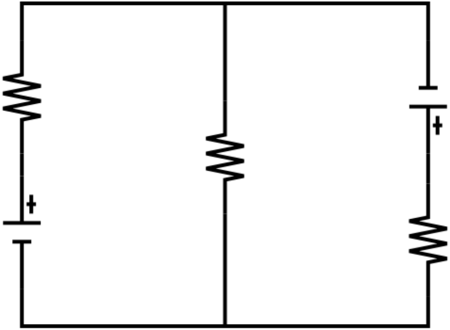
where A is the coefficient matrix and X is the vector of the currents. (Because of the two types of equations used, B is a vector of voltages and zeros.)

If a solution exists, it may be found by

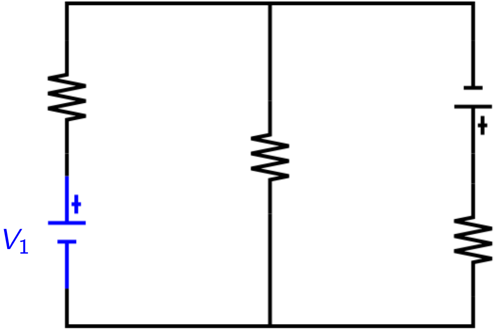
$$X = A^{-1}B$$

where A^{-1} is the inverse of the coefficient matrix. (A solution will not exist if you have, for instance, erroneously connected two different voltage sources in parallel.)

So far no mention has been made of the *dimension* of the coefficient matrix. Clearly, to be invertible A must be square, however you will probably have many more equations than currents. This is because the equations are *not* all **linearly independent**. In other words, some of the equations are redundant, and will have to be eliminated for the system to be solvable.



Label all V's and R's.



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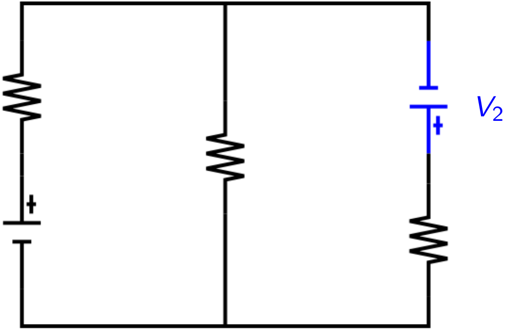

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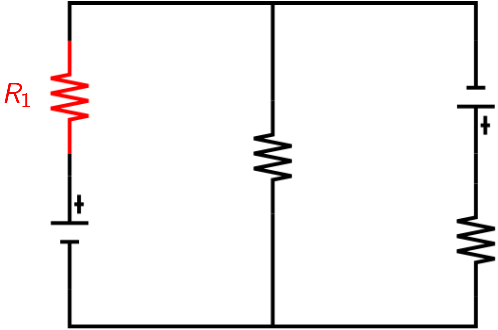

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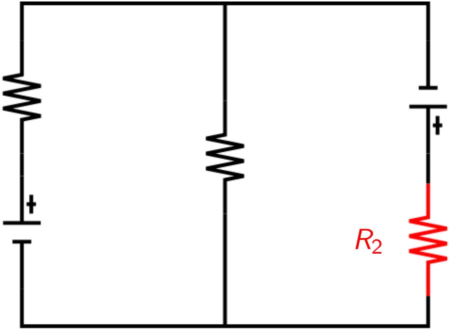

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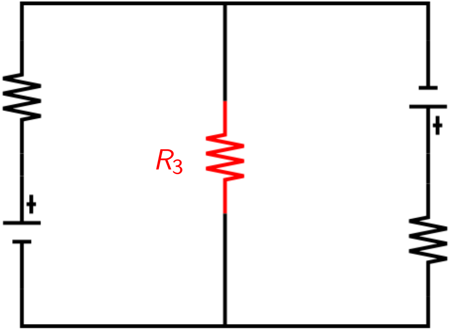

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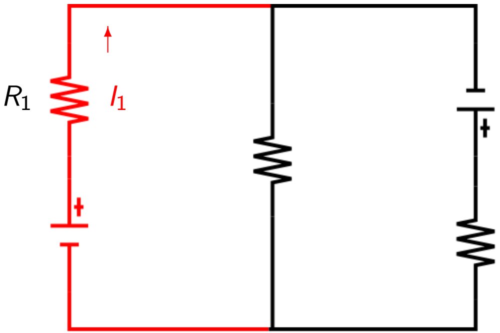
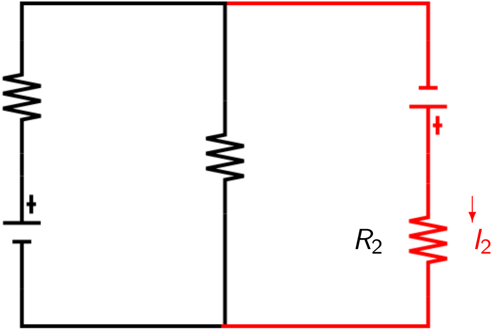


Label all I's to go with R's and pick directions for I's.



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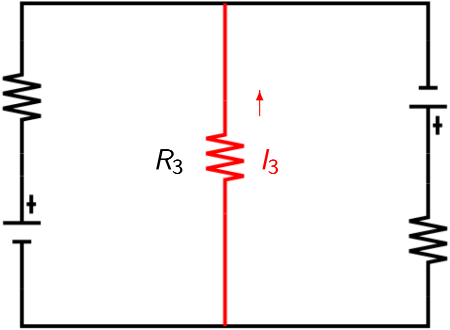

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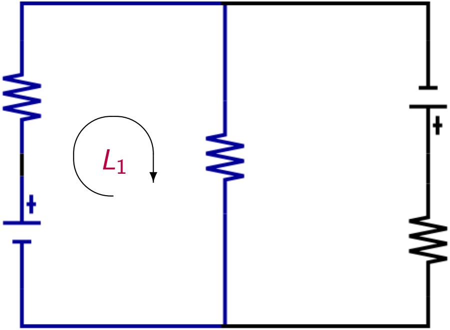
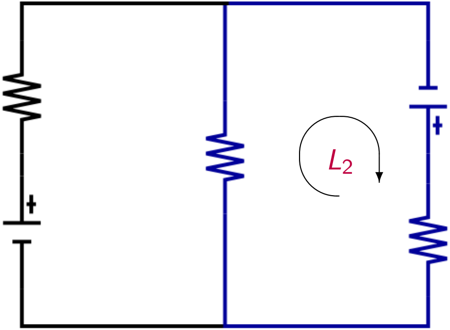


Label loops.



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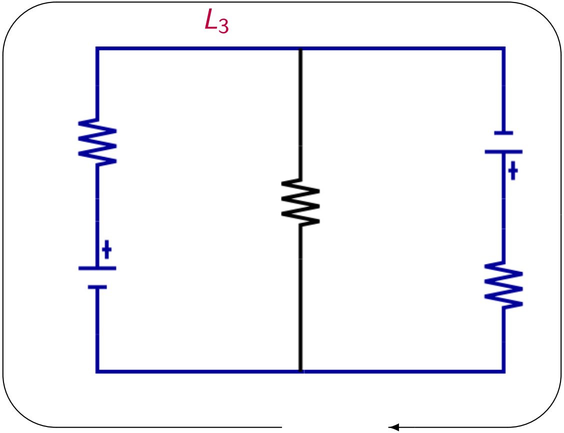

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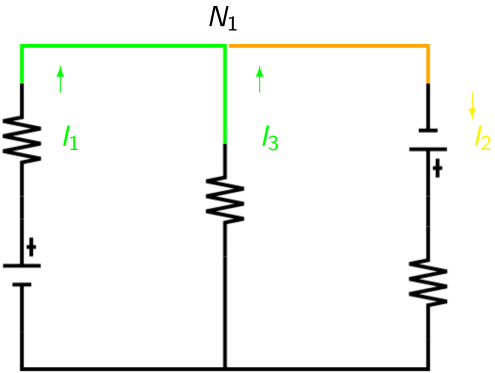
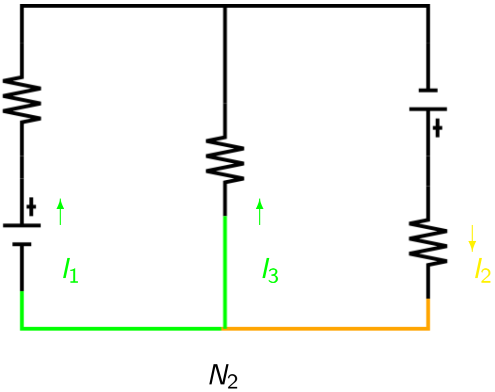


Label nodes.



Label nodes.

Making loop equations

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1. Start at node, go around to first component.

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For each loop:

1. Start at node, go around to first component.
2. If component is a battery, count the voltage as *positive* if you come to the '-' terminal first.
3. If component is a resistor, count the IR as *positive* if you come to the resistor going *against* the current.
4. Repeat for all components until you are back at the node.

Making loop equations

For each loop:

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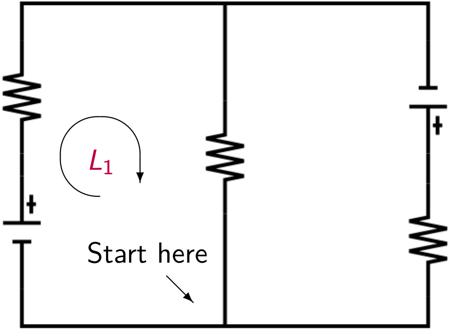
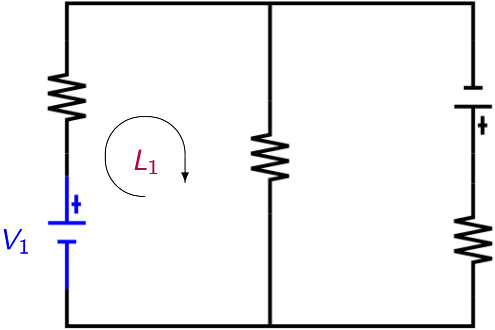
2. If component is a battery, count the voltage as *positive* if you come to the '-' terminal first.
3. If component is a resistor, count the IR as *positive* if you come to the resistor going *against* the current.
4. Repeat for all components until you are back at the node.
5. Set the sum of all of the contributions from 2 and 3 to zero.

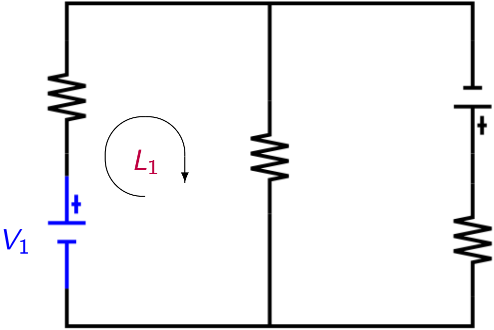
Here's how the equation for Loop 1 is created.



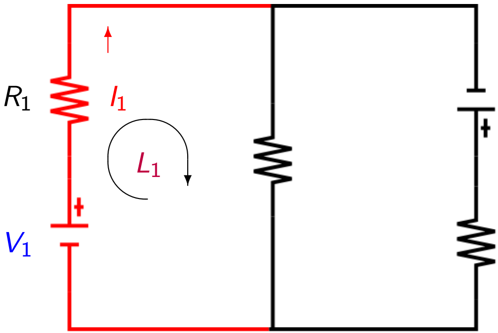
Start at the lower right corner of this loop.



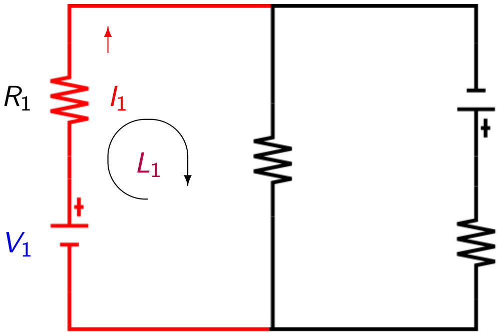
V_1 is the first thing we encounter.



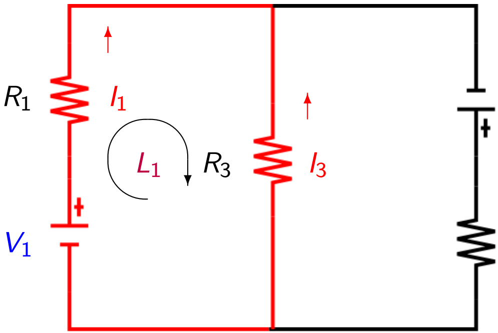
V_1 is *positive* since we hit the negative terminal first.



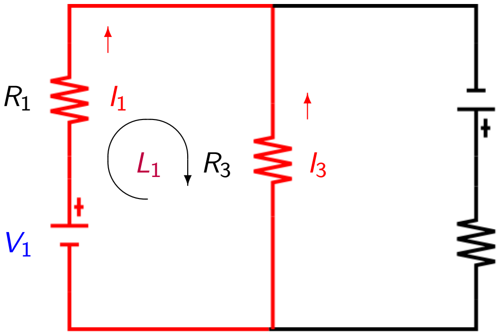
R_1 is the next thing we encounter.



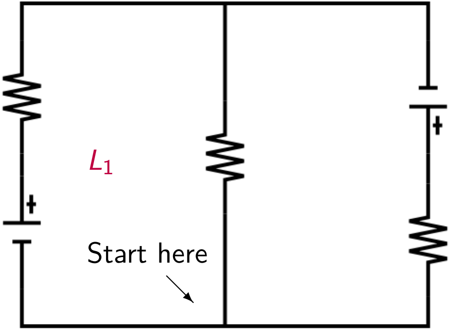
$I_1 R_1$ is *negative* since the loop direction *matches* the current direction.

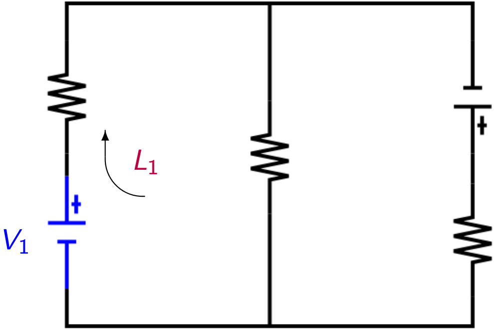


R_3 is next.

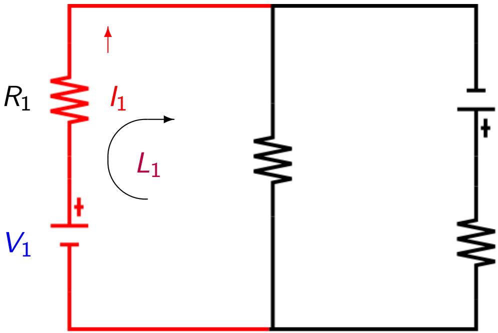


$I_3 R_3$ is *positive* since the loop direction *opposes* the current direction.

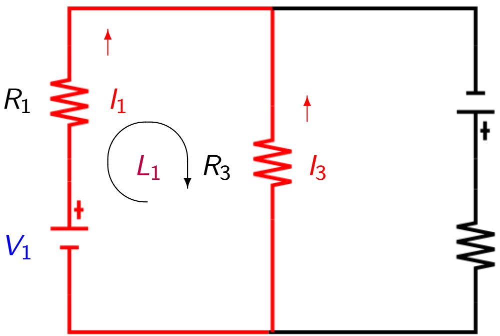

L_1
Start here



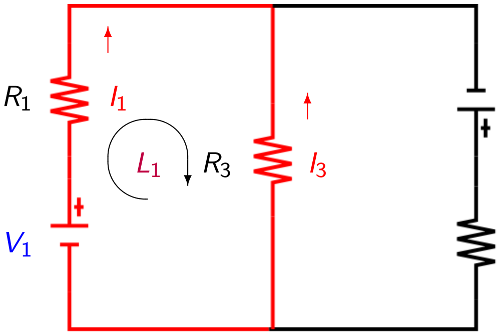
$+V_1$



$+V_1 - I_1 R_1$



$$+V_1 - I_1 R_1 + I_3 R_3$$



$$+V_1 - I_1 R_1 + I_3 R_3 = 0$$

Thus for the example above:

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Loop 1 (starting at Node 1)

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Loop 2 (starting at Node 2)

$$V_2 - I_2 R_2 - I_3 R_3 = 0$$

Thus for the example above:
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Loop 2 (starting at Node 2)

$$V_2 - I_2 R_2 - I_3 R_3 = 0$$

Loop 3 (starting at Node 1)

$$V_1 - I_1 R_1 + V_2 - I_2 R_2 = 0$$

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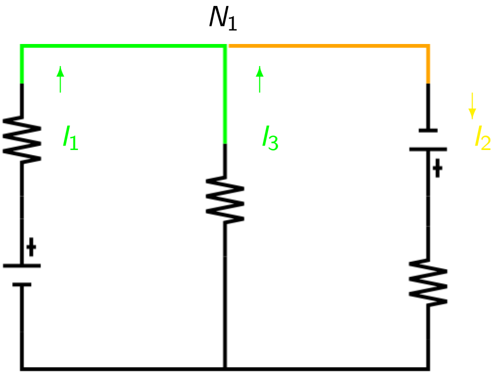

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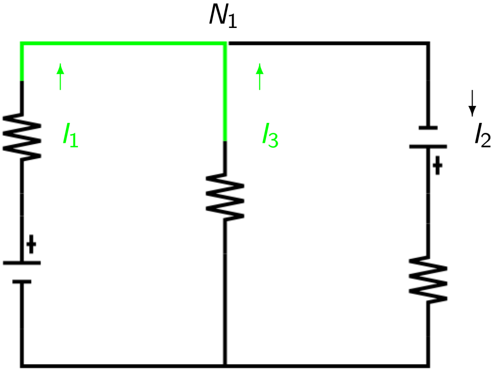
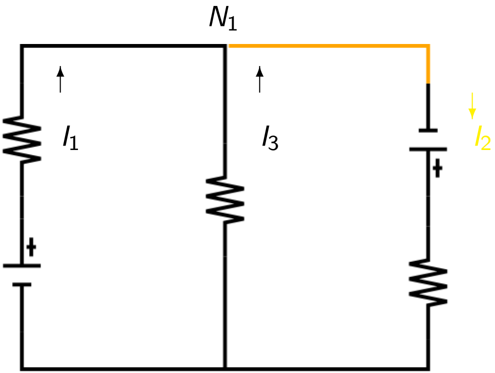
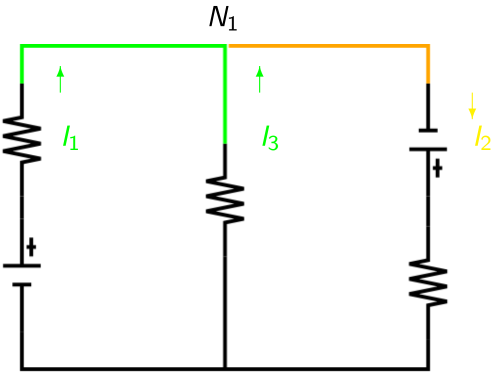


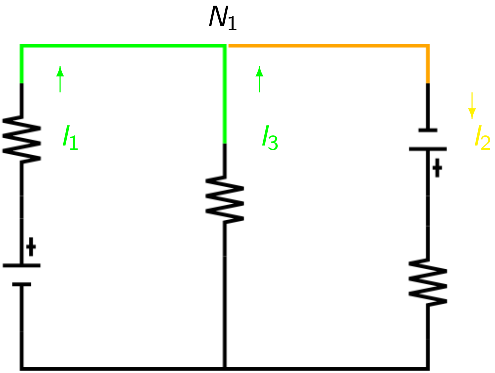
I_1 and I_3 go *into* Node 1.



I_2 comes *out of* Node 1.



$$I_1 + I_3 = I_2$$



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Node 2

$$I_1 + I_3 - I_2 = 0$$

Set up a system of equations.

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$$
\begin{array}{rcccccc}
Loop\ 1: & -I_1R_1 & & +I_3R_3 & = & -V_1 \\
Loop\ 2: & & -I_2R_2 & -I_3R_3 & = & -V_2 \\
Loop\ 3: & -I_1R_1 & -I_2R_2 & & = & -V_1 - V_2 \\
Node\ 1: & -I_1 & +I_2 & -I_3 & = & 0 \\
Node\ 2: & +I_1 & -I_2 & +I_3 & = & 0
\end{array}
$$

Create matrices.

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We can rewrite the above system in matrix form as

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$$\begin{pmatrix} -R_1 & 0 & +R_3 \\ 0 & -R_2 & -R_3 \\ -R_1 & -R_2 & 0 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} -V_1 \\ -V_2 \\ -V_1 - V_2 \\ 0 \\ 0 \end{pmatrix}$$

So clearly if we let

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$$A = \begin{pmatrix} -R_1 & 0 & +R_3 \\ 0 & -R_2 & -R_3 \\ -R_1 & -R_2 & 0 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{pmatrix}$$

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$$A = \begin{pmatrix} -R_1 & 0 & +R_3 \\ 0 & -R_2 & -R_3 \\ -R_1 & -R_2 & 0 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{pmatrix}$$

$$X = \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}$$

So clearly if we let

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$$X = \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}$$

$$B = \begin{pmatrix} -V_1 \\ -V_2 \\ -V_1 - V_2 \\ 0 \\ 0 \end{pmatrix}$$

Then

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$$AX = B$$

Reduce coefficient matrix to square, since it will not be invertible otherwise.

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$$A = \begin{pmatrix} -R_1 & 0 & +R_3 \\ 0 & -R_2 & -R_3 \\ -R_1 & -R_2 & 0 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{pmatrix}$$

Here's the matrix

$$A = \begin{pmatrix} -R_1 & 0 & +R_3 \\ 0 & -R_2 & -R_3 \\ -R_1 & -R_2 & 0 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{pmatrix}$$

Notice row 1

$$A = \begin{pmatrix} -R_1 & 0 & +R_3 \\ 0 & -R_2 & -R_3 \\ -R_1 & -R_2 & 0 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{pmatrix}$$

Notice row 1 +row 2

$$A = \begin{pmatrix} -R_1 & 0 & +R_3 \\ 0 & -R_2 & -R_3 \\ -R_1 & -R_2 & 0 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{pmatrix}$$

Notice row 1 +row 2 equal row 3, so we can get rid of row 3

$$A = \begin{pmatrix} -R_1 & 0 & +R_3 \\ 0 & -R_2 & -R_3 \\ -R_1 & -R_2 & 0 \\ \color{red}{-1} & \color{red}{+1} & \color{red}{-1} \\ +1 & -1 & +1 \end{pmatrix}$$

Also, row 4

$$A = \begin{pmatrix} -R_1 & 0 & +R_3 \\ 0 & -R_2 & -R_3 \\ -R_1 & -R_2 & 0 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{pmatrix}$$

Also, row 4 = -row 5, so get rid of row 5

So we are left with

$$\mathcal{A} = \begin{pmatrix} -R_1 & 0 & +R_3 \\ 0 & -R_2 & -R_3 \\ -1 & +1 & -1 \end{pmatrix}$$

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$$\mathcal{A} = \begin{pmatrix} -R_1 & 0 & +R_3 \\ 0 & -R_2 & -R_3 \\ -1 & +1 & -1 \end{pmatrix}$$

and

$$\mathcal{B} = \begin{pmatrix} -V_1 \\ -V_2 \\ 0 \end{pmatrix}$$

after adjustment

Now

$$X = \mathcal{A}^{-1}\mathcal{B}$$

Now

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which gives

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which gives

$$X = \frac{1}{R_2 R_1 + R_2 R_3 + R_1 R_3}\begin{pmatrix} R_3 V_1 + R_3 V_2 + R_2 V_1 \\ R_3 V_1 + R_1 V_2 + R_3 V_2 \\ R_1 V_2 - R_2 V_1 \end{pmatrix}$$

Now

$$X = \mathcal{A}^{-1}\mathcal{B}$$

which gives

$$X = \frac{1}{R_2 R_1 + R_2 R_3 + R_1 R_3} \begin{pmatrix} R_3 V_1 + R_3 V_2 + R_2 V_1 \\ R_3 V_1 + R_1 V_2 + R_3 V_2 \\ R_1 V_2 - R_2 V_1 \end{pmatrix}$$

Notice that, depending on the voltages involved, some of the currents could be negative. In this case, it simply means that the actual current goes in the opposite direction from what you had chosen.