Uncertain Results

1. Operations with Uncertainties

When numbers, some or all of which are approximate, are combined by addition, subtraction, multiplication, or division, the uncertainty in the results due to the uncertainties in the data is given by the range of possible calculated values based on the range of possible data values. For instance, if we have two numbers with uncertainties, such as $x = 2.5 \pm 1.1$ and $y = 32.0 \pm 0.2$, then what that means is that x can be as small as 1.4 or as big as 3.6, while y can be as small as 31.8 or as big as 32.2 so adding them can give a result $x+y$ which can be as small as 32.8 or as big as 35.2, so that the uncertainty in the answer is the sum of the two uncertainties. If we call the uncertainties in x and y Δx and Δy, then we can illustrate as follows:

1.1 Adding

$$x \pm \Delta x = \ \ 2.5 \pm 0.1$$
$$+\ y \pm \Delta y = 32.0 \pm 0.2$$
$$(x + y) \pm (?) = 34.5 \pm 0.3$$

$$x \pm \Delta x + y \pm \Delta y = (x + y) \pm (\Delta x + \Delta y)$$

Thus

$$\Delta(x + y) = \Delta x + \Delta y$$

Thus $x+y$ can be between 34.2 and 34.8, as above.

1.2 Subtracting

If we subtract two numbers, the same sort of thing happens.

$$x \pm \Delta x = 45.3 \pm 0.4$$
$$-y \pm \Delta y = 18.6 \pm 0.3$$

$$(x - y) \pm (?) = 63.9 \pm 0.7 = (x - y) \pm (\Delta x + \Delta y)$$

Note that we still add the uncertainties, even though we subtract the quantities.

1.3 Multiplying

Multiplication and division are a little different. If a block of wood is found to have a mass of 1.00 ± 0.03 kg and a volume of 0.020 ± 0.001 m^3, then the nominal value of the density is $1.00\text{kg}/0.020\text{m}^3 = 50.0\text{kg/m}^3$ and the uncertainty in its density may be determined as follows:

The mass given above indicates the mass is known to be greater than or equal to 0.97 kg, while the volume is known to be less than or equal to 0.021 m^3. Thus, the minimum density of the block is given by $0.97\text{kg}/0.021\text{m}^3 = 46.2\text{kg/m}^3$. Similarly, the mass is known to be less than or equal to 1.03 kg, while the volume is known to be greater than or equal to 0.019 m^3. Thus, the maximum density of the block is given by $1.03\text{kg}/0.019\text{m}^3 = 54\text{kg/m}^3$.

Notice that the above calculations do not give a symmetric range of uncertainties about the nominal value. This complicates matters, but if uncertainties are small compared to the quantities involved, the range is approximately symmetric and may be estimated as follows:

$$x \pm \Delta x = 8.2 \pm 0.3 = 8.2 \pm (0.3/8.2 \times 100\%)$$
$$\times \quad y \pm \Delta y = 7.1 \pm 0.2 = 7.1 \pm (0.2/7.1 \times 100\%)$$

$$(x \times y) \pm (?) = 58 \pm ((0.3/8.2 + 0.2/7.1) \times 100\%)$$
$$= 58 \pm (2+2)$$
$$= 58 \pm 4$$
$$= (x \times y) \pm (x \times y) \times (\frac{\Delta X}{X} + \frac{\Delta Y}{Y})$$

So rather than adding absolute uncertainties, we add relative or percent uncertainty to the operation of multiplying.

$$x \times y = 54 \pm 4$$

If you're a purist, or if the uncertainties are not small, then the uncertainty in the density can then be estimated in two obvious ways;

1. the greater of the two differences between the maximum and minimum and the accepted values
2. (or the maximum and minimum values can both be quoted, which is more precise, but can be cumbersome if subsequent calculations are necessary.)

(In the previous example, the first method would give an uncertainty of 4.2 kg/m^3.)

1.4 Dividing

Division is similar to multiplication, as subtraction was similar to addition.

$$x \pm \Delta x = 7.6 \pm 0.8 = 7.6 \pm (0.8/7.6 \times 100\%)$$
$$y \pm \Delta y = 2.3 \pm 0.1 = 2.3 \pm (0.1/2.3 \times 100\%)$$
$$(x \div y) \pm (?) = 3.3 \pm 3.3 \times (0.1 + 0.04)$$
$$= 3.3 \pm 0.3$$

As a similar result for multiplying, we add relative or percent uncertainty to the operation of dividing.

2. Determining Uncertainties in Functions of Quantities by Inspection

Note: In the following section and elsewhere in the manual, the notation Δx is used to mean "the uncertainty in x".

Suppose we measure the diameter of a marble, d, with an uncertainty Δd, then quantities such as the volume derived from d will also have an uncertainty. Specifically,

$$\Delta V = V(d + \Delta d) - V(d)$$

In this equation we could call $V(d)$ the nominal value of V, $V(d + \Delta d)$ the maximum value of V, and $V(d - \Delta d)$ the minimum value of V. Since V and d are always positive, this will be true.

Note that the above calculation of ΔV may not be exactly the same as $\Delta V = V(d) - V(d - \Delta d)$, which is equally valid, but if the uncertainties are small, they should be similar. Specifically, since

$$V = \frac{4}{3}\pi(\frac{d}{2})^3$$

then

$$\Delta V = \frac{4}{3}\pi(\frac{d+\Delta d}{2})^3 - \frac{4}{3}\pi(\frac{d}{2})^3$$

In general,

$$\Delta f(x) = |f(x + \Delta x) - f(x)|$$

where the absolute value signs indicate that the result should always be given as a positive quantity. Another way of writing this is

$$\Delta f(x \pm \Delta x) = |f'(x)|\Delta x$$

(This form makes it easier to adapt for functions of more than one variable.)
The following section describes how to determine the uncertainty in quantities of this sort. (The first approach is sort of \brute force"; later a rule will be given which will usually be easier to use.)
Actually, these two forms are equal if $\Delta d << d$. You can derive this using the binomial approximation, which simply means multiplying it out and discarding and terms with two or more Δ terms multiplied together; for instance $\Delta A \Delta B \cong 0$
Using the above example,

$$V' = 2\pi(\frac{d}{2})^2 = \frac{\pi}{2}d^2$$

and so

$$\Delta V = \frac{\pi}{2}d^2\,\Delta d$$

3. Determining Uncertainties by Trial and Error

For a function $f(x; y)$, the uncertainty in f will be given by the biggest of

$$|f(x + \Delta x; y + \Delta y) - f(x; y)|$$

or

$$|f(x - \Delta x; y + \Delta y) - f(x; y)|$$

or

$$|f(x + \Delta x; y - \Delta y) - f(x; y)|$$

or

$$|(x - \Delta x; y - \Delta y) - f(x; y)|$$

Note that for each variable with an uncertainty, the number of possibilities doubles. In most cases, common sense will tell you which one is going to be the important one, but things like the sign of numbers involved, etc. will matter a lot! For example, if you are adding two positive quantities, then the first or fourth above will give the same (correct) answer. However, if one quantity is negative, then the second and third will be correct.
For example, if we were trying to find the uncertainty in

$$\rho = \frac{m}{v}$$

then algebraically we would get

$$\Delta\rho = \rho(\frac{\Delta m}{m} + \frac{\Delta v}{v})$$

and by inspection

$$\Delta\rho = \left| \frac{m + \Delta m}{v - \Delta v} - \frac{m}{v} \right|$$

since to make the biggest difference in a fraction we increase the numerator and decrease the denominator. (Note that for some complicated equations, this second method may be easier than the first. Also, the two may give slightly different answers. This will be most pronounced when the uncertainty in a quantity is close to the size of the quantity itself.)

The advantage of knowing this method is that it always works. Sometimes it may be easier to go through this approach than to do all of the algebra needed for a complicated expression, especially if common sense makes it easy to see which combination of signs gives the correct answer.

4. Determining Uncertainties Algebraically

Usually, the uncertainty in results can be calculated as in the following examples (if the percentage uncertainties in the data are small):

(a) $\Delta(A + B) = (\Delta A + \Delta B)$

(b) $\Delta(A - B) = (\Delta A + \Delta B)$

(c) $\Delta(A \times B) = |AB| \left(\left| \frac{\Delta A}{A} \right| + \left| \frac{\Delta B}{B} \right| \right)$

(d) $\Delta\left(\frac{A}{B}\right) = \left| \frac{A}{B} \right| \left(\left| \frac{\Delta A}{A} \right| + \left| \frac{\Delta B}{B} \right| \right)$

(e) $\Delta f(A \pm \Delta A) = |f'(A)|\Delta A$

Note that the first two rules above always hold true.

To summarize, when adding or subtracting, you add absolute uncertainties.

When multiplying or dividing, you add percent or relative uncertainties.

Note that for the last rule above that angles and their uncertainties must be in radians for the differentiation to be correct! (In the examples above, absolute value signs were omitted since all positive quantities were used.)

(Some specific uncertainty results can be found in Appendix I.)

5. Simplifying Uncertainties

Uncertainty calculations can get quite involved if there are several quantities involved. However, since uncertainties are usually only carried to one or two significant figures at most, there is little value in carrying uncertainties through calculations if they do not contribute significantly to the total.

You do not need to carry uncertainties through if they do not contribute more than 10% of the total uncertainty. (However be sure to explain when you do this.)

Note that this shows a difference between doing calculations by hand versus using a spreadsheet. If you are doing calculations by hand, it makes sense to drop insignificant uncertainties like this. However, if you are using a spreadsheet in order to allow you to change the data and recalculate, it may be worth carrying all uncertainties through in case some of them may be more significant for different data.

When a scientist performs an experiment, it is crucial to determine whether or not the results make sense. In other words, do any calculated quantities fall within a "reasonable" range?

The reason for doing calculations with uncertainties is so that uncertainties in final answers can be obtained. If, for instance, a physical constant was measured, the calculated uncertainty determines the range around the calculated value in which one would expect to find the “theoretical” value.

If the theoretical value falls within this range, then we say that our results agree with the theory within our experimental uncertainty.

Mathematically, if two quantities a and b, with uncertainties Δa and Δb are compared, they can be considered to agree within their uncertainties if

$$|a - b| \leq \Delta a + \Delta b$$

If we need to compare 3 or more values this becomes more complex. (An “ideal” value can be considered as one with zero uncertainty.)

If two quantities agree within experimental error, this means that the discrepancy between experiment and theory can be readily accounted for on the basis of measurement uncertainties which are known. If the theoretical value does not fall within this range, then we say that our results do not agree with the theory within experimental uncertainty. In this situation, we cannot account for the discrepancy on the basis of measurement uncertainties alone, and so some other factors must be responsible.

If two numbers do not agree within experimental error, then the percentage difference between the experimental and theoretical values must be calculated as follows:

$$\text{Percent Diffence} = \left| \frac{Theretical - Experimental}{Theretical} \right| \times 100\%$$

(Often instead of comparing an experimental value to a theoretical one, we are asked to test a law such as the Conservation of Energy. In this case, what we must do is to compare the initial and final energies of the system in the manner just outlined. If the values agree, then we can say that energy was conserved, and if the values don't agree then it wasn't.)

Remember: Only calculate the percent difference if your results do not agree within experimental error.

6. Significant Figures in Final Results

Always express final answers with absolute uncertainties rather than percent uncertainties. Also, always quote final answers with one digit of uncertainty, and round the answers so that the least significant digit quoted is the uncertain one. This follows the same rule for significant figures in measured values.