Repeated Measurements
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Overview

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- how to determine uncertainties in repeated measurements

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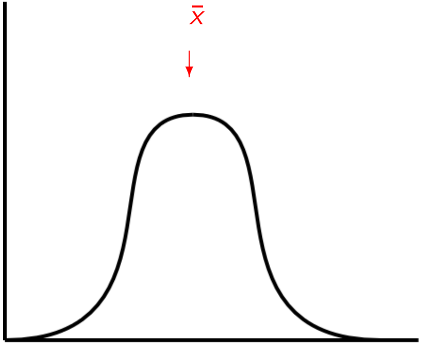
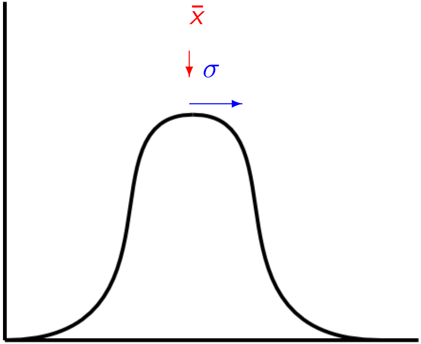
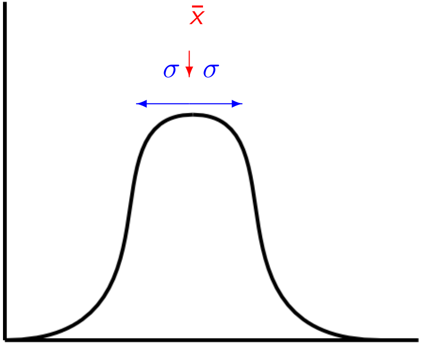
- how to determine uncertainties in repeated measurements
- how to determine the optimal number of measurements to take



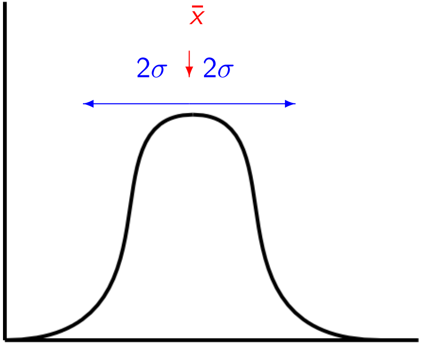
- $\bar{x}$ is the *mean* (average)



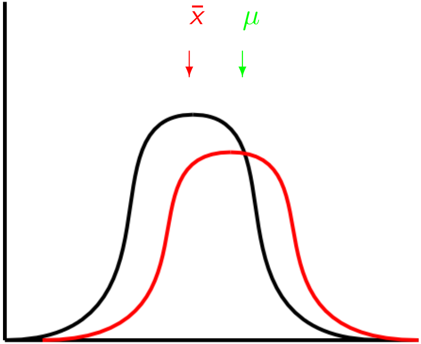
- σ is the *standard deviation*



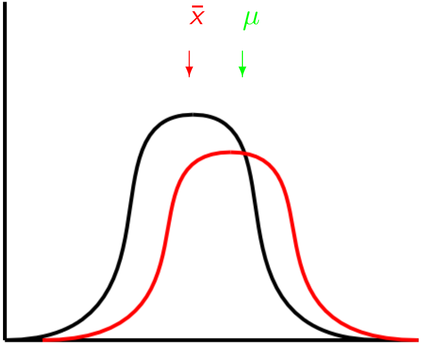
- About $2/3$ of the measurements should fall within $\bar{x} \pm \sigma$



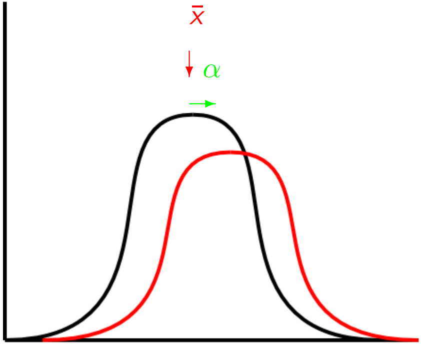
- About 95% of the measurements should fall within $\bar{x} \pm 2\sigma$



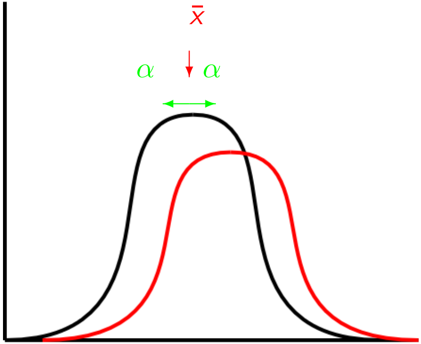
- μ is the *population* mean; i.e. what we'd get with *lots* of measurements



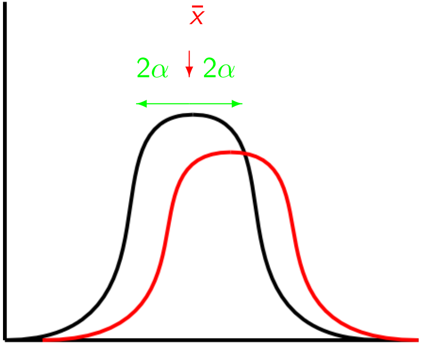
- The distance between $\bar{x}$ and μ will get smaller as you take more measurements



- The *standard deviation of the mean*, α, is the distance from $\bar{x}$ within which we expect to find μ



- About 2/3 of the time, μ will be within $\bar{x} \pm \alpha$



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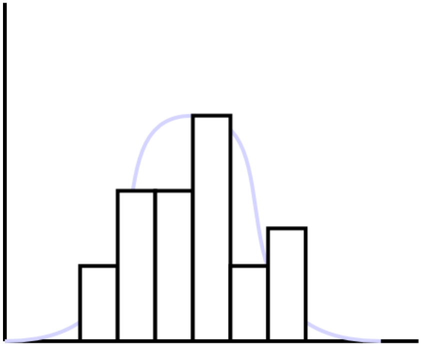
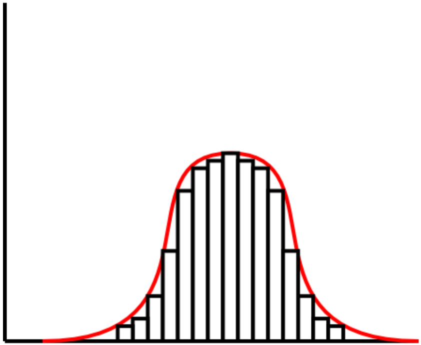


With a small sample, the shape may be only approximately normal, and the mean will be approximate



With a lot of measurements, the shape should become more normal (Gaussian), and the mean will be more reliable

For example, say you are in a class of about 200 students. On a test, you get 74%. You want to figure out the class average. You try to do it by asking more and more people for their marks.

In the figures that follow, the marks have been divided into *bins* that are 5 marks wide. So there is a bin for marks from 50 to 55, a bin for marks from 55 to 60, etc.

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Approximate values of $\bar{x}$, σ, and α are highlighted.

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person	mark
1	73.6
2	76.9
3	66.9
4	78.8
5	64.9
6	65.5
7	72.4
8	69.8
9	69.2
10	71.6

We can rearrange the numbers in increasing order so that we can make a **histogram** of the values.

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person	mark	bin
5	64.9	60-65
6	65.5	65-70
3	66.9	65-70
9	69.2	65-70
8	69.8	65-70
10	71.6	70-75
7	72.4	70-75
1	73.6	70-75
2	76.9	75-80
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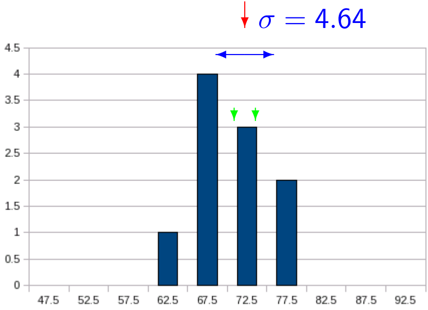
$\bar{x} = 70.96$

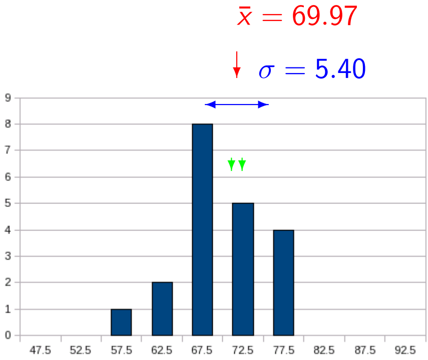
$\sigma = 4.64$



$N = 10$

$\sigma = 4.64$; 95% of the marks should be between 62 and 80

$\alpha = 1.47$; 95% chance class average between 68 and 74



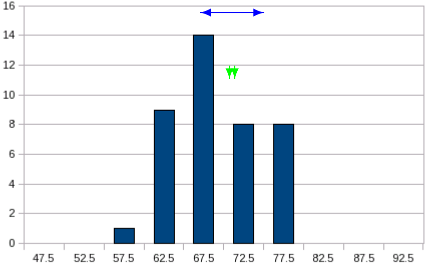
$N = 20$

$\sigma = 5.40$; 95% of the marks should be between 59 and 81

$\alpha = 1.21$; 95% chance class average between 67.6 and 72.4

$\bar{x} = 69.13$

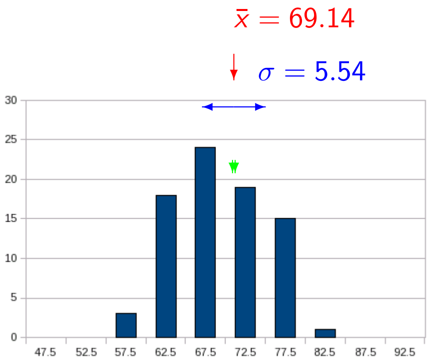
$\sigma = 5.44$



$N = 40$

$\sigma = 5.44$; 95% of the marks should be between 58 and 80

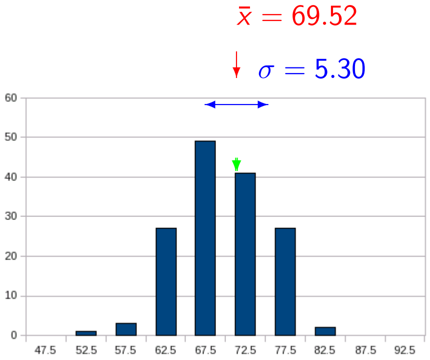
$\alpha = 0.86$; 95% chance class average is between 67.3 and 70.9



$N = 80$

$\sigma = 5.54$; 95% of the marks should be between 58 and 80

$\alpha = 0.62$; 95% chance class average is between 67.9 and 70.3



$N = 150$

$\sigma = 5.30$; 95% of the marks should be between 59 and 80

$\alpha = 0.43$; 95% chance class average is between 68.7 and 70.3

Calculating statistics

The equations for these quantities are:

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  $\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}} = \frac{1}{\sqrt{n-1}}\sqrt{\sum_{i=1}^{n} x_i^2 - \frac{(\sum_{i=1}^{n} x_i)^2}{n}}$

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The standard deviation will always be bigger than the **standard deviation of the mean**

Calculating standard deviation

i	x_i	x_i^2
1	1.1	1.21
n	$\sum x_i$	$\sum x_i^2$
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i	x_i	x_i^2
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2	1.4	1.96
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2	2.5	3.17

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$$= \frac{1}{\sqrt{4-1}}\sqrt{6.3 - \frac{(5.0)^2}{4}} = 0.13$$

Calculating standard deviation of the mean

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$$\alpha = \frac{\sigma}{\sqrt{n}} = \frac{0.13}{\sqrt{4}} = 0.064$$

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We can *assume* it is 0.1 since that's the smallest gap between x values.

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The uncertainty in the average is the bigger of these two quantities; the precision measure and the standard deviation of the mean.

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$= 1.2 \pm 0.1$ after rounding

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  $= \left(\frac{0.13}{0.01}\right)^2$
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  So we could take about another 164 measurements.

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When we have the optimal number of measurements, (i.e. precision measure *equals* the standard deviation of the mean), we would have to do *both* in order to improve the experiment.

Recap

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4. How to improve the experiment depends on which quantity is bigger; the standard deviation of the mean or the precision measure.