PC132 Lab Manual
Wilfrid Laurier University

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Chapter 1

Goals for PC132 Labs

The labs in PC132 will build on the skills developed in the labs for PC131. Emphasis will be on learning how to do experimental science, and on how to communicate well, rather than on illustrating particular physical laws.

For these reasons, the goals of the *labs* in this course will fall into these general areas:

- introduce *graphing and graphical analysis*, which will include

    - linearizing equations
    - error bars on graphs
    - least squares fitting

- familiarize you with using the library for *research*

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Chapter 2

Instructions for PC132 Labs

Students will be divided into sections, each of which will be supervised by a lab supervisor and a demonstrator. This lab supervisor should be informed of any reason for absence, such as illness, as soon as possible. (If the student knows of a potential absence in advance, then the lab supervisor should be informed in advance.) A student should provide a doctor's certificate for absence due to illness. Missed labs will normally have to be made up, and usually this will be scheduled as soon as possible after the lab which was missed *while the equipment is still set up for the experiment in question.*

It is up to the student to read over any theory for each experiment and understand the procedures and do any required preparation before the laboratory session begins. This may at times require more time outside the lab than the time spent in the lab.

You will be informed by the lab instructor of the location for submission of your reports during your first laboratory period. This report will usually be graded and returned to you by the next session. The demonstrator who marked a particular lab will be identified, and any questions about marking should first be directed to that demonstrator. *Such questions must be directed to the marker within one week of the lab being returned to the student if any additional marks are requested.*

2.1 Expectations

As a student in university, there are certain things expected of you. Some of them are as follows:

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- You are expected to *come to the lab prepared*. This means first of all that you will ensure that you have all of the information you need to do the labs. After you have been told what lab you will be doing, you should read it ahead and be clear on what it requires. You should bring the lab manual, lecture notes, etc. with you to every lab. (*Of course you will be on time* so you do not miss important information and instructions.)

- You are expected to *be organized* This includes recording raw data with sufficient information so that you can understand it, keeping proper backups of data, reports, etc., hanging on to previous reports, and so on. It also means starting work early so there is enough time to clarify points, write up your report and hand it in on time.

- You are expected to *be adaptable and use common sense*. In labs it is often necessary to change certain details (eg. component values or procedures) at lab time from what is written in the manual. You should be alert to changes, and think rationally about those changes and react accordingly.

- You are expected to *value the time of instructors and lab demonstrators*. This means that you make use of the lab time when it is scheduled, and try to make it as productive as possible. This means NOT arriving late or leaving early and then seeking help at other times for what you missed.

- You are expected to *act on feedback from instructors, markers, etc.* If you get something wrong, find out how to do it right and do so.

- You are expected to *use all of the resources at your disposal*. This includes everything in the lab manual, textbooks for other related courses, the library, etc.

- You are expected to *collect your own data*. This means that you perform experiments with your partner and *no one else*. If, *due to an emergency*, you are *forced* to use someone else's data, you *must* explain why you did so and explain *whose* data you used. Otherwise, you are committing *plagiarism*.

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- You are expected to *do your own work.* This means that you prepare your reports with *no one else.* If you ask someone else for advice about something in the lab, make sure that anything you write down is based on your *own* understanding. If you are basically regurgitating someone else's ideas, even in your own words, you are committing *plagiarism.* (See the next point.)

- You are expected to *understand your own report.* If you discuss ideas with other people, even your partner, do not use those ideas in your report unless you have adopted them yourself. You are responsible for all of the information in your report.

- You are expected to *be professional* about your work. This means meeting deadlines, understanding and meeting requirements for labs, reports, etc. This means *doing what should be done*, rather than what you think you can get away with. This means proofreading reports for spelling, grammar, etc. before handing them in.

- You are expected to *actively participate* in your own education. This means that in the lab, you do not leave tasks to your partner because you do not understand them. This means that you try and learn *how* and *why* to do something, rather than merely finding out the *result* of doing something.

2.2   Workload

Even though the labs are each only worth part of your course mark, the amount of work involved is probably disproportionately higher than for assignments, etc. Since most of the "hands-on" portion of your education will occur in the labs, this should not be surprising. (*Note: skipping lectures or labs to study for tests is a very bad idea.* Good time management is a much better idea.)

2.3   Administration

1. Students are advised to have a binder to contain all lab manual sections (if using the printed manual) and all lab reports which have been returned. (A 3 hole punch will be in the lab.)

2. Templates will be used in each experiment as follows:

    (a) The data in the template must be checked by the demonstrator before students leave the lab.

    (b) No more than 3 people can use one set of data. If equipment is tight groups will have to split up. (i.e. Only as many people as fit the designated places for names on a template may use the same data.)

    (c) The template *must* be included with lab handed in. If you use a spreadsheet to collect your data, rather than the printed data, print off the original spreadsheet page with your original data.

    (d) If a student misses a lab, and if space permits (decided by the lab supervisor) the student may do the lab in another section the same week without penalty. (However the due date is still for their own section.) In that case the section they record on the template should be where the experiment was *done*, not where the student normally belongs.

3. Answers to *pre-lab* questions are to be brought to the lab. They are to be handed in *at the beginning of the lab*.

4. Answers to the *in-lab* questions are to be handed in at the end of the lab.

5. Students are to make notes about pre- and in-lab question answers and keep them in their binders so that the points raised can be discussed in their reports. Marks for answers to questions will be separate from marks for the lab. For people who have missed the lab without a doctor's note and have not made up the lab, these marks will be forfeit. The points raised in the answers will still be expected to be addressed in the lab report.

6. *Post-lab* questions are to be answered after calculations. For *exercises*, answers are to be handed in and the mark for post-lab questions will be part of the exercise mark. For *labs*, the answers should be included as part of the lab report. The mark for post-lab questions will be part of the mark for the lab report.

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7. *Pre-lab* tasks are to be performed before the lab, and will get checked off during the first few minutes of the lab.

8. *In-lab* tasks are to be performed during the lab, and must get checked off before you leave the lab.

9. *Post-lab* tasks are to be performed after the lab, and will get checked off during the first few minutes of the next lab period.

10. Labs handed in after the due date incur a penalty of 5% per day late to a maximum penalty of 50%. After the reports for an experiment have been returned, any late reports submitted for that experiment cannot receive a grade higher than the lowest mark from that lab section for the reports which were submitted on time.

2.4 Plagiarism

11. Plagiarism includes the following:

    - Identical or nearly identical wording in any block of text.
    - Identical formatting of lists, calculations, derivations, etc. which *suggests* a file was copied.

12. You will get one warning the first time plagiarism is suspected. After this any suspected plagiarism will be forwarded directly to the course instructor. With the warning you will get a zero on the relevant section(s) of the lab report. If you wish to appeal this, you will have to discuss it with the lab supervisor and the course instructor.

13. If there is a suspected case of plagiarism involving a lab report of yours, it does not matter whether yours is the original or the copy. The sanctions are the same.

2.5 Calculation of marks

The calculation of the final lab mark is explained in Appendix D, *"Marking Scheme"*.

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Chapter 3

Presenting Data in Reports

3.1 Introduction

Previously, you have prepared lab reports primarily with a *marker* in mind. However you should be starting to gear your reports to a more general *reader*. How this will change your report should become clear as you read the following:

3.1.1 Purpose of a report

The goal of presenting a report is to *inform*, not to *impress*. That means that, on the one hand, you don't want to fill space with drivel just to make the reader think you know something, (it's not likely to work), but on the other hand, at times it may be helpful to repeat a piece of useful information two or three times in a report to save the reader having to flip back and forth. Individual sections should be as self–contained as possible, so that a reader is not normally forced to hunt for pertinent facts all through the report.

3.1.2 Structuring a report

In some labs, lab templates may have been used to organize reports in a very standard way to give some uniformity to the reports. Now you will not have that order imposed, and so you will have to structure your own reports so that they are understandable. Part of what this will require is for you to put in enough *"English glue"* to make the report easy to read, even (especially!) for someone who does not have the lab manual at hand.

3.1.3   Moving from lab notes to a report

When handling data, either to analyze it or to present it, it is important to make a distinction between between *utility* and *clarity*. In other words, how you set up a spreadsheet to analyze data or graph it may *not* be the way that you should set it up for someone else to look at. Similarly, showing the output block from a least squares fit reflects whether you did it correctly, but what you should present is the *meaningful* results of the fit, not every bit of output. (If it needs to be included for a marker, put it in an appendix so that it's there, but does not hurt the flow of the report.) Following are some guidelines for presenting data for the *reader*, not for the *writer* or the *marker*.

3.2   Text

- Grammar and spelling count!

- All numerical quantities must include uncertainties!

- In the text of a report, all symbols should be explained, especially if they are non–standard (for instance if you use "w" instead of "ω"). For instance *"w is the angular frequency in rads/sec.")* Units should be given for each quantity as well.

- The report should have brief descriptions of procedures, etc., so that a person not following the manual can still make sense of the data. If you are following a manual, you need not go into great detail, but the significance of parameters stated, etc. should be explained; eg. *"current was measured by calculating the voltage across resistor R_M"*

- Quotes, standard values, etc. should be foot-noted and referenced.

- Derivations may be done by hand (if long), but if you are using a word processor, this is a chance to learn more features if you use it to do at least the short derivations. Make sure the symbols you use in derivations match the symbols you use in the text. (See above example with w and ω.)

- Watch for similar or duplicate symbols; eg. e, the base for natural logarithms, and e the charge on an electron (or in the above example

where you might have both w and ω). If you have to use two symbols like this in one report, change one (and define it!) or change both. (For example, use q for the charge on an electron, or replace e^x with $\exp(x)$.)

- Quantities which would normally be expressed using scientific notation with uncertainties should usually be presented in the standard form; eg. $(2.3 \pm 0.2) \times 10^{-6} m$.

> *The purpose of scientific notation is to remove* **placeholder** *zeroes, either before or after the decimal point, from a number. Thus it often does NOT make sense in numbers in the range from about* $0.1 \rightarrow 99$*, where there are no placeholder zeroes.*

- All results should be given the correct number of significant figures; i.e. one or at most two significant figures for uncertainties, and quantities rounded so least significant digit is in the same place as the least significant digit of uncertainty.

3.3 Tables

- Always include tables of *raw data*, even if you need to modify the data to plot a graph. That way if you make a mistake in calculations, it will be possible to correct later.

- Tables must have boxes around them and lines separating columns, etc. i.e. unstructured spreadsheets are *not* OK.

- Any data which will be plotted in a graph should be shown in a table with the same units and uncertainties as on the graph.

- No table should be split by a page break; if necessary make it into two separate tables.

- All tables need names and numbers such as "Table 1" (which should be referenced in the report), and *meaningful* labels which match the text (or explanations of how the labels correspond to the quantities in the text). Table 3.1has two obvious problems; the column labels are somewhat cryptic, and much data is redundant.

Vx	dVx	Ix	dIx	Used (y/n)
0.10	0.02	1.5	0.3	y
0.19	0.02	1.3	0.3	y
0.24	0.02	1.2	0.3	y
0.41	0.02	0.8	0.3	n

Table 3.1: One way to do it

In Table 3.2, some of this is changed. This is much less "busy", and more descriptive.

Voltage	Current
0.10	1.5
0.19	1.3
0.24	1.2
0.41	$0.8^{\dagger}$

All voltages $\pm$ 0.02 volts.
All currents $\pm$ 0.3 amps.
† point not used in fit.

Table 3.2: More concise way

Least squares fit output can be somewhat confusing; indicate which points were used (if not all, as in the above example), the fit equation, and the parameters calculated as well as their standard errors. Be sure to include the proper units for both. A table may not even be a good way to give these.

3.4   Graphs

- *All graphs must include error bars!* If error bars in one or both dimensions are too small to be seen on a graph, then a note should be made on the graph to indicate this.

- Titles should be *descriptive*; i.e. they should give pertinent information which is not elsewhere on the graph.

- Graphs can be annotated with fit results so that by looking at the graph the reader can see fit results (with uncertainties, of course). Make sure

to mark which points were used for the fit (if not all). Keep in mind the above rules about symbols.

- Fit results should be given in terms of actual graphical quantities, not x and y. For instance, *"Slope is 4.5 ± 0.2 N/kg"; y-intercept is $3.1 \pm 0.1N$"*, as opposed to *"y=4.5x+3.1"* which lacks uncertainties, units, and *relevance*.

- Put units on each axis, and either use a grid for both dimensions, or else none at all. (A horizontal–only grid looks kind of odd.)

- Graphs should be in the specified orientation. *NOTE: a graph of y versus x means y is on the vertical axis and x is on the horizontal axis.*

3.5   Linearizations

- Always include the original (i.e. non-linearized) equation(s) as well as the linearized one(s).

3.6   Least Squares Fits

- Always plot data (with error bars) before fitting to see that points make sense. (Make sure error bars are correct!)

- Clearly identify data used in fit (if not all points are used).

- Give results meaningful names, such as "slope", "standard error in slope", etc.

- Include *units* for slope, y-intercept, etc.

- Show the fit line on the graph with the data.

- Identify whether the graph shows "small" or "large" scatter, and then according to that identification, do whichever of these is appropriate:

  - Perform fit in such a way as to get standard errors in both y-intercept and slope.

– Find maximum and minimum slopes (if they exist) and show them on the graph with the data.

- Determine the uncertainties in the slope and the y–intercept from the result above.

- If you would have expected either the slope and the y–intercept to be zero, and it isn't, then suggest why that might be so.

3.7 Other

Here are a couple of final tips.

> Printing out a spreadsheet with formulas shown does not count as showing your calculations.

The reader should not have to be familiar with spreadsheet syntax to make sense of results.

> You must discuss at least one source of systematic error in your report, even if you reject it as insignificant, in order to indicate how it would affect the results.

Chapter 4

Graphs and Graphical Analysis

4.1 Introduction

One of the purposes of a scientific report is to present numerical information, i.e. data and calculated results, in concise and meaningful ways. As with other parts of the report, the goal is to make the report as self-explanatory as possible. Ideally a person unfamiliar with the experiment should be able to understand the report without having to read the lab manual. (In your case, the reader can be assumed to be familiar with the general procedure of the experiment, but should not be expected to be intimately familiar with experiment-specific symbols. For instance, if you must measure the diameter of an object in the lab, and use the symbol d for it, be sure to state what d represents the first time it is used.)

A physical law is a mathematical relationship between measurable quantities, as has been stated earlier. A graph is a visual representation of such a relationship. In other words, a graph is always a representation of a particular mathematical relationship between the variables on the two axes; usually these relationships are made to be functions.

As a representation of how data are related, a graph will usually contain both data points and a **fitted curve** showing the function which the data should follow. (The term "curve" may include a straight line. In fact, it is often easiest to interpret results when an equation has been **linearized** so that the graph should be a straight line. Linearization will be discussed in Chapter 6, *"Exercise on Linearizing Equations"* .)

With single values which are measured or calculated, when there is an

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"expected" value, then uncertainties are used to determine how well the experimental value matches the expected value. For a set of data which should fit an equation, it is necessary to see how all points match the function. This is done using **error bars**, which will be discussed later. In essence, error bars allow one to observe how well each data point fits the curve or line on the graph. When parameters of an equation, such as the slope and y-intercept of a straight line, are determined from the data, (as will be discussed later), then those parameters will have uncertainties which represent the range of values needed to make all of the data points fit the curve.

4.2 Graphing

4.2.1 Data Tables

Often the data which is collected in an experiment is in a different form than that which must be plotted on a graph. (For instance, masses are measured but a graph requires weights.) In this case, the data which is to be plotted should be in a data table of its own. This is to make it easy for a reader to compare each point in the data table with its corresponding point on the graph. The data table should include the size of error bars for each point, in each dimension. Units in the table should be the same as on the graph.

Any graph must be plotted from data, which should be presented in tables. Tables should

- have ruled lines outside and separating columns, etc. to make it neat and easy to read

- have meaningful title and column headings

- not be split up by page breaks (i.e. unless a table is bigger than a single page, it should all fit on one page.)

- have a number associated with it (such as *"Table 1"*) for reference elsewhere in the report, and a name, (such as *"Steel Ball Rolling down Incline"*) which makes it self–explanatory

- include the information required for any numerical data, i.e. units, uncertainties, etc.

A sample is shown in Table 4.1.

i	x_i (cm)	Δx_i (cm)	t_i (s)	Δt_i (s)
1	0.40	0.03	0.0	0.1
2	0.77	0.04	2.0	0.1
3	1.35	0.04	2.7	0.1

Table 4.1: Position versus Time for Cart

4.2.2 Parts of a Graph

1. Title

   The title of a graph should make the graph somewhat self–explanatory aside from the lab. Something like "y vs. x" may be correct but redundant and useless if the person viewing the graph can read. "Object in Free Fall" would be more helpful as the reader may be able to figure out the *significance* of the graph herself.

2. Axis Labels

   As above, "m" and "l" are not as useful as "added mass (m) in grams", and "length of spring (l) in cm". In this case the words are meaningful, while the symbols are still shown to make it easy to find them in equations. Units *must* be included with axis labels.

3. Axis Scales

   The following 3 points are pertinent if you are plotting graphs "by hand". If you use a spreadsheet, these things are usually taken care of automatically.

   (a) Always choose the scales of the axes so that the data points will be spread out over as much of the plotting area as possible.

   (b) Choose the scales in a convenient manner. Scales that are easy to work with are to be preferred over scales such as ones where every small division corresponds to 0.3967 volts, for example. A better choice in such a case would be either 0.25, 0.50, or perhaps even 0.4 volts per division, the decision of which would be determined by the previous constraint. If you have *discrete*, i.e. integer, values on one axis, do not use scientific notation to represent those values.

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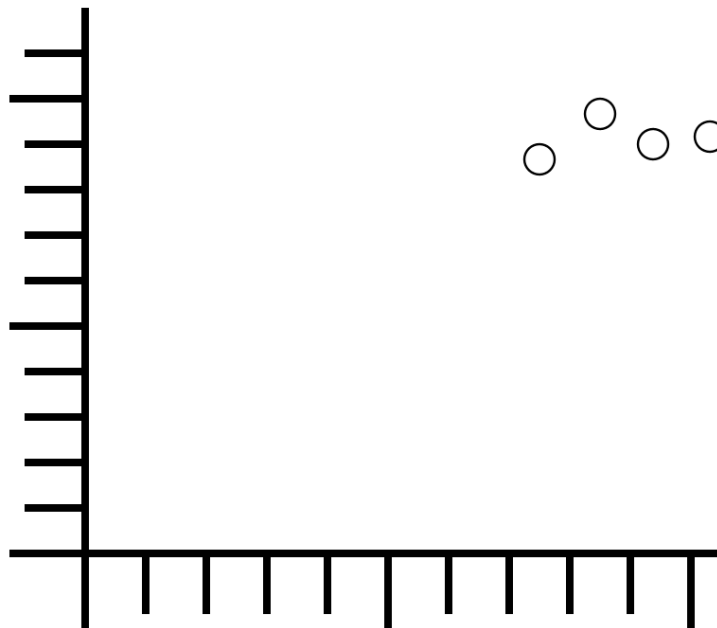
Figure 4.1: Wrong: Data, (not empty space), should fill most of the graph

(c) (0,0) does *not* have to be on graph – *data* should cover more than 1/4 of the graph area; if you need to extrapolate, do it numerically.

4. Plotting Points

Often, results obtained from graphs are slightly suspicious due to the simple fact that the experimenter has incorrectly plotted data points. If plotting by hand, be careful about this. Data points *must be* fitted with *error bars* to show uncertainties present in the data values. If the uncertainties in either or both dimensions are too small to show up on a particular graph, a note to that effect should be made on the graph so that the reader is aware of that fact.

> *Do not connect the points like a dot-to-dot drawing!*

5. Points for Slope

   In calculating parameters from a graph, such as the slope, points on a line should be chosen which are *not* data points, *even if data points appear to fall directly on the line*; failure to follow this rule makes the actual line drawn irrelevant and misleading.

   When plotting points for the slope, a different symbol should be used from that used for data points to avoid confusion. The co–ordinates of these data points should be shown near the point as well for the reader's information. If one uses graph paper with a small enough grid, it may be possible to choose points for the slope which fall on the intersection of grid lines which simplifies the process of determining their co–ordinates. Of course, points for the slope should always be chosen as far apart as possible to minimize errors in calculation.

6. Error Bars

   Data points *must be* fitted with *error bars* to show uncertainties present in the data values. If the uncertainties in either or both dimensions are too small to show up on a particular graph, a note to that effect should be made on the graph so that the reader is aware of that fact. Uncertainties in quantities plotted on a graph are shown by error bars. Figure 4.2 shows a point with its error bars. The range of possible values for the data point in question actually includes any point bounded by the rectangle whose edges fall on the error bars. The size of the error bars is given by the uncertainties in both coordinates. (Actually, the point's true value is most likely to fall within the ellipse whose extents fall on the error bars. This is because it is unlikely that the x and y measurements are both in error by the maximum amount at the same time.) In fact, error bars may be in one or both directions, and they may even be different in the positive and negative directions.

Is the origin a data point?

Sometimes an experiment produces a graph which is expected to go through $(0, 0)$. In this case, whether to include the origin as a data point or not arises. There is a basic rule: *Include $(0, 0)$ as a data point only if you have measured it (like any other data point).* Often a graph which is *expected* to go through the origin will not do so due to some experimental factor which
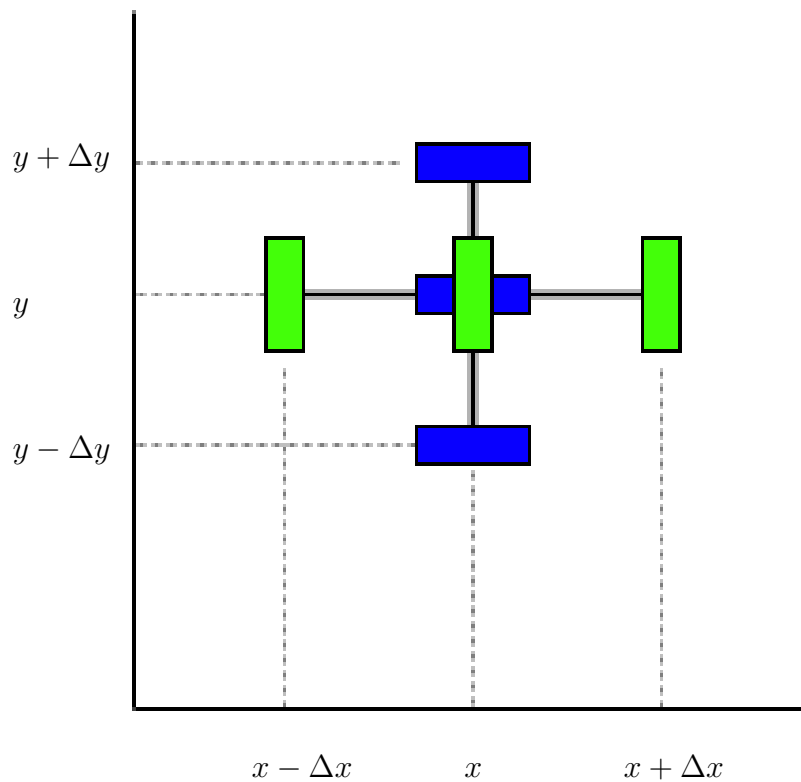
Figure 4.2: Point with error bars

was not considered in the derivation of the equation. It is important that the graph show what really happens so that these unconsidered factors will in fact be noticed and adjusted for. This brings up a second rule: *If the origin is a data point, it is no more "sacred" than any other data point.* In other words, don't force the graph through $(0,0)$ any more than you would through any other point. Doing a least squares fit will protect you from this temptation.
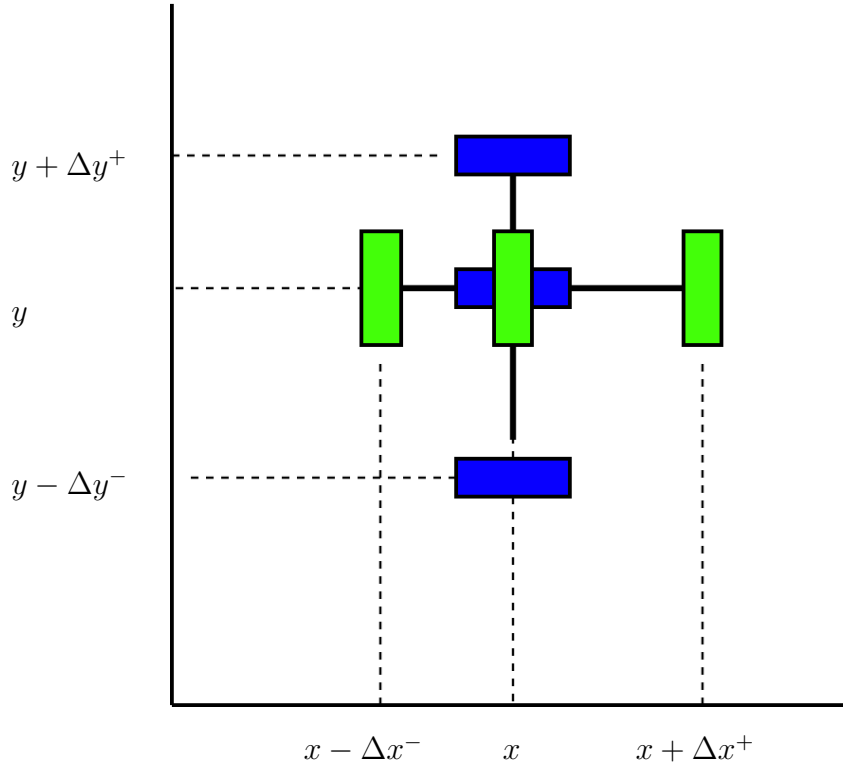
Figure 4.3: Graph with unequal error bars in positive and negative directions

4.3 Graphical Analysis

Usually the point of graphing data is to determine parameters of the mathematical relationship between the two quantities. For instance, when plotting a straight line graph, the slope and y–intercept are the parameters which describe that relationship.

Note that the slope and y–intercept and their uncertainties should have units. The units of the y–intercept should be the same as the y variable, and the slope should have units of

$$[slope] = \frac{[y]}{[x]}$$
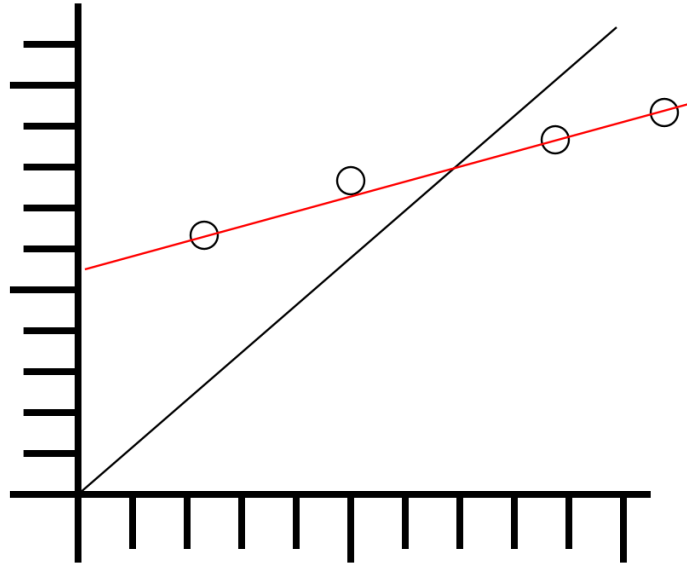
Figure 4.4: A line through the origin is not always the best fit

4.4   Linearizing Equations

In many cases, the mathematical model you are testing will suggest how the data should be plotted. A great deal of simplification is achieved if you can *linearize* your graph, i.e., choose the information to be plotted in such a way as to produce a straight line. (This is discussed in Chapter 6, *"Exercise on Linearizing Equations"* .) For example, suppose a model suggests that the relationship between two parameters is

$$z = Ke^{-\lambda t}$$

where K and λ are constants. If a graph of the natural logarithm of z is plotted as a function of t, a straight line given by

$$\ln z = \ln K - \lambda t$$

will be obtained. The parameters K and λ will be much easier to determine graphically in such a case.

In particular, if we substitute $y = \ln z$ and $x = t$ in the above equation, and if the slope and y–intercept are measured to be, respectively, m and b, then it should be clear that

$$m = -\lambda$$

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and

$$b = \ln K$$

4.5 Curve Fitting

Always draw smooth curves through your data points, unless you have reason to believe that a discontinuity in slope at some point is genuine.

> *Your graphs should not look like a dot–to–dot drawing.*
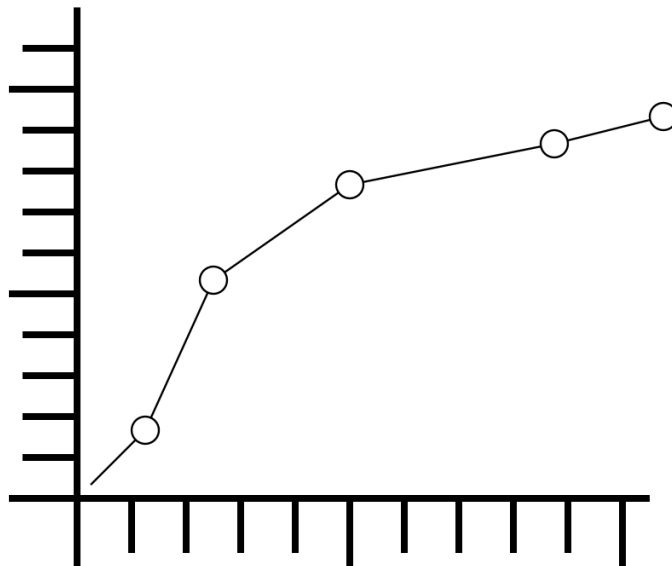
Figure 4.5: Wrong: Graphs should not look like dot-to-dot drawings

If you are plotting the points using the computer, draw the curve by hand if necessary to avoid this problem. However, do not fit data to a curve with no physical significance simply so that all of the points fit.

> *Do not use an arbitrary function just because it goes through all the data points!*
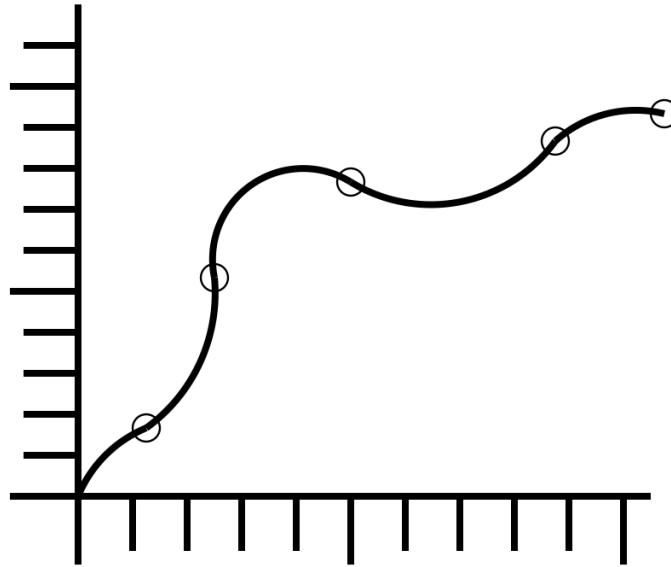
Figure 4.6: Wrong: Graphs should not have meaningless curves just to fit the data

Note that unless a set of data exactly *fits a curve, choosing* a *curve of "best fit" is somewhat arbitrary.* (For example, consider 4 data points at (-1,1), (1,1), (1,-1) and (-1,-1). What line fits these points best?)

Usually, going "by eye" is as good as anything; the advantage to a method such as the *least squares fit* is that it is easily automated, and is generally reliable.

If plotting by eye, one should observe that the line of best fit will usually have an equal number of points above and below it. As well, as a rule, there should not be several points at either end of the graph on the *same* side of the curve. (If this is the case, the curve can be adjusted to avoid this.)

Determining the y-intercept is easy if it is shown on the graph. However if it isn't, you can determine it from the points you used for the slope. If

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

and

$$y = mx + b$$

for any points on the line, including (x_1, y_1) and (x_2, y_2) then

$$y_2 = mx_2 + b$$

so

$$b = y_2 - mx_2$$

and finally

$$b = y_2 - \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x_2$$

4.6   Least Squares Fitting

Least **S**quares **F**itting is a procedure for numerically determining the equation of a curve which "best approximates" the data being plotted. If we wish to fit a straight line to data in the form

$$y = mx + b$$

then the least squares fit gives values for b, the y-intercept, and m, the slope, as follows:[1]

$$b = \frac{\left(\sum y_i\right)\left(\sum x_i^2\right) - \left(\sum x_i\right)\left(\sum x_i y_i\right)}{N\left(\sum x_i^2\right) - \left(\sum x_i\right)^2} \qquad (4.3)$$

and

$$m = \frac{N\left(\sum x_i y_i\right) - \left(\sum x_i\right)\left(\sum y_i\right)}{N\left(\sum x_i^2\right) - \left(\sum x_i\right)^2} \qquad (4.4)$$

(Note: You do not need to calculate uncertainties for m and b during least squares fit calculations like this. Uncertainties in m and b will be dealt with later.)

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[1]You may notice that a particular quantity comes up a lot. It is

$$N\left(\sum x_i^2\right) - \left(\sum x_i\right)^2 \qquad (4.1)$$

It only takes a couple of lines of algebra to show that this equals

$$N\left(N - 1\right)\sigma_x{}^2 \qquad (4.2)$$

where $\sigma_x{}^2$ is the sample standard deviation of the x values.

If you are interested, an appendix contains a derivation of the least squares fit. In any case you may use the result above.

One important concept which will come up later is that of **degrees of freedom**, which is simply the number which is the difference between the number of data points, (N above), and the number of parameters being determined by the fit, (2 for a straight line). Thus, for a linear fit, the number of degrees of freedom, ν, is given by:

$$\nu = N - 2 \tag{4.5}$$

4.6.1 Correlation coefficient

Equation 4.6 gives the square of the **Pearson product-moment correlation coefficient**, which we will refer to simply as the *correlation coefficient*.[2]

$$R^2 = \frac{\left(N \sum x_i y_i - (\sum x_i)(\sum y_i)\right)^2}{\left(N \sum x_i^2 - (\sum x_i)^2\right)\left(N \sum y_i^2 - (\sum y_i)^2\right)} \tag{4.6}$$

The correlation coefficient, R, is a number which has a value between -1 and +1, where a value of -1 indicates a perfect negative correlation, +1 indicates a perfect positive correlation, and a value of zero indicates no correlation. Thus R^2 is a value between zero and 1 indicating just the *strength* of a correlation. The closer R^2 is to one, the stronger the correlation between two variables. To put it another way, the closer it is to one the better one variable can be used as a *predictor* of the other.

4.7 Uncertainties in Graphical Quantities

After the slope and intercept have been calculated, their associated errors are calculated in one of two ways depending on the data. (This is analogous to the idea that the uncertainty in the average is the bigger of the standard deviation of the mean and the uncertainty in the individual values.) The two possible cases are outlined below.

---

[2]As long as we're dealing with a linear fit, this is the quantity that would commonly be used.

> Unless the points fit exactly on a straight line, any graph with big enough error bars will fit the first case, and any graph with small enough error bars will fit the second. It is the relative size of the error bars which determines which case it is.

4.7.1 Small Scatter of Data

If the **scatter** in the data points is small, a straight line which passes through *every* error bar on the graph can be found, as shown in Figure 4.7. This indicates that the uncertainties in your results are *primarily* due to the uncertainties of the measuring instruments used.

The slope and intercept can be found graphically, by eye or using the least squares fit method.
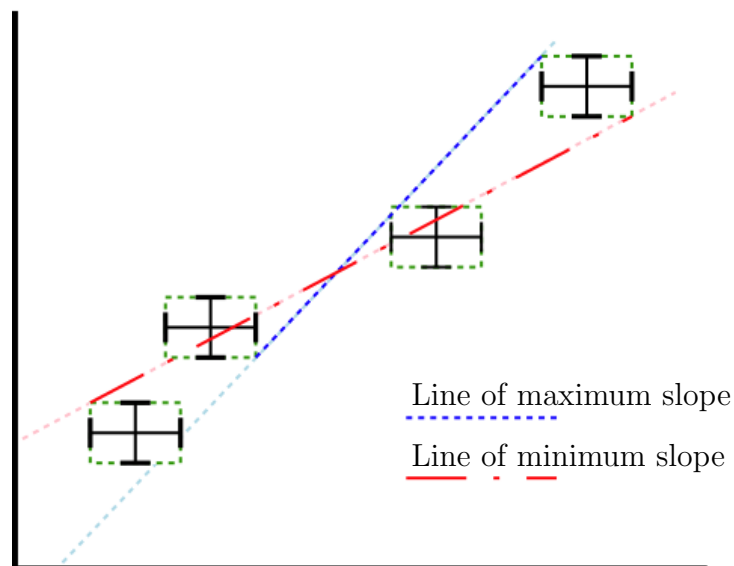
Figure 4.7: Small Scatter of Data Points

To obtain error estimates in these quantities, one draws two lines: a line with the *maximum* slope passing through *all* the error bars; and the line with the *minimum* slope passing through *all* the error bars. These extremes will determine the required uncertainties in the slope and intercept. *(In this*

graph and the one following, boxes have been drawn around each point and its error bars to indicate the "uncertainty region" around each point. These would not usually be on a graph, but they are shown here for illustration.)

The points for the maximum and minimum slope will not always be the endpoints on the graph. Also, the data points providing the endpoints for the two lines will not usually be the same for both. If the maximum and minimum slope are not symmetric about the average, you can calculate

$$\Delta m \approx \frac{m_{max} - m_{min}}{2}$$

and

$$\Delta b \approx \frac{b_{max} - b_{min}}{2}$$

$(x_2 - \Delta x_2, \, y_2 + \Delta y_2)$

$(x_2 + \Delta x_2, \, y_2 - \Delta y_2)$

$(x_1 - \Delta x_1, \, y_1 + \Delta y_1)$

For positive slope

$(x_1 + \Delta x_1, \, y_1 - \Delta y_1)$

$(x_1 + \Delta x_1, \, y_1 + \Delta y_1)$

$(x_2 + \Delta x_2, \, y_2 + \Delta y_2)$

$(x_1 - \Delta x_1, \, y_1 - \Delta y_1)$

For negative slope
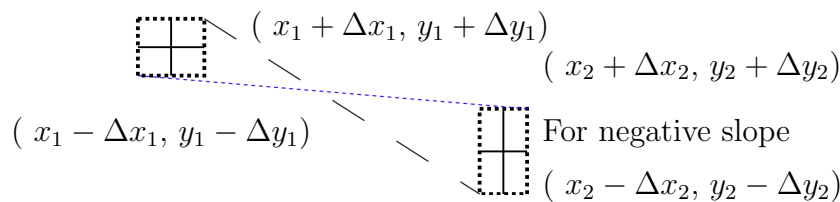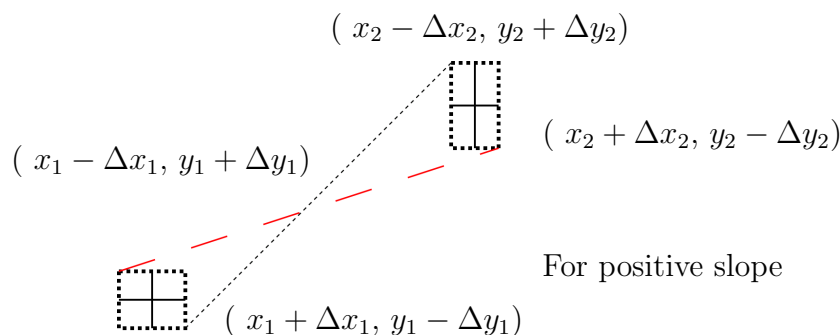
$(x_2 - \Delta x_2, \, y_2 - \Delta y_2)$

Figure 4.8: Maximum and Minimum Slope Coordinates from a Point

If we label two points x_1 and x_2, where $x_1 < x_2$, then we can see from

Figure 4.8 that the steepest line which touches the error bars for both x_1 and x_2 is the line between $(x_1 + \Delta x_1, y_1 - \Delta y_1)$ and $(x_2 - \Delta x_2, y_2 + \Delta y_2)$. The slope of this line will then be

$$m_{max} = \frac{(y_2 + \Delta y_2) - (y_1 - \Delta y_1)}{(x_2 - \Delta x_2) - (x_1 + \Delta x_1)} \tag{4.7}$$

and then the y-intercept is given by

$$b_{min} = (y_1 - \Delta y_1) - m_{max}(x_1 + \Delta x_1) = (y_2 + \Delta y_2) - m_{max}(x_2 - \Delta x_2) \tag{4.8}$$

Similarly the line with the least slope which touches the error bars for both x_1 and x_2 is the line between $(x_1 - \Delta x_1, y_1 + \Delta y_1)$ and $(x_2 + \Delta x_2, y_2 - \Delta y_2)$. The slope of this line will then be

$$m_{min} = \frac{(y_2 - \Delta y_2) - (y_1 + \Delta y_1)}{(x_2 + \Delta x_2) - (x_1 - \Delta x_1)} \tag{4.9}$$

and then the y-intercept is given by

$$b_{max} = (y_1 + \Delta y_1) - m_{min}(x_1 - \Delta x_1) = (y_2 - \Delta y_2) - m_{min}(x_2 + \Delta x_2) \tag{4.10}$$

The case for a negative slope is shown in Figure 4.8; the analysis is left to the student.

> *The points for the maximum and minimum slope will not always be the endpoints on the graph.*

4.7.2 Large Scatter of Data

Often, you will not be able to find a line which crosses *every* error bar, as with the data in Figure 4.9, and you will have to resort to the numerical method below. In this case, the uncertainties in your graphical results are *primarily* due to the random variations in the data.

Once these values for the slope and intercept are determined, the sum of squares error, S is computed. For the linear case, S can be shown to have a value of

$$S = \sum y_i^2 - m \sum x_i y_i - b \sum y_i \tag{4.11}$$

In order to estimate the uncertainty in each parameter, the standard deviation σ is computed from

$$\sigma = \sqrt{\frac{S}{N - 2}} \tag{4.12}$$
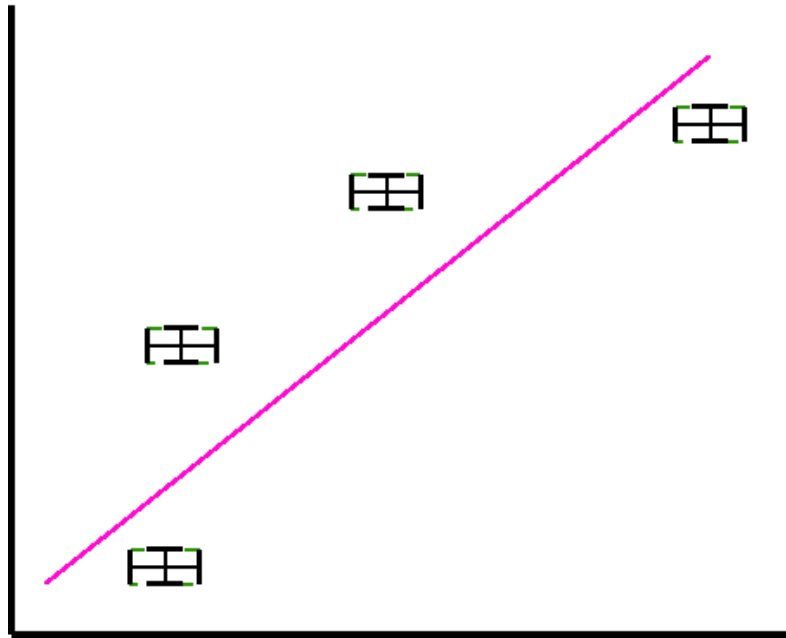
Figure 4.9: Large Scatter of Data

where $N - 2$ is the number of degrees of freedom mentioned earlier. (Often the symbol ν is used for degrees of freedom.) The **standard error** (i.e. uncertainty) in the intercept is

$$\sigma_b = \sigma \sqrt{\frac{\sum x_i^2}{N\left(\sum x_i^2\right) - \left(\sum x_i\right)^2}} \tag{4.13}$$

and the standard error (uncertainty) in the slope is

$$\sigma_m = \sigma \sqrt{\frac{N}{N\left(\sum x_i^2\right) - \left(\sum x_i\right)^2}} \tag{4.14}$$

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As long as our data fit this second case, then we can use σ_b and σ_m as the uncertainties in the y–intercept and the slope respectively, and use the symbols Δb and Δm instead. Keep in mind, however, that if our data fit the first case, then these terms are not interchangeable. Note that the uncertainties in the slope and y–intercept should have the same units as the slope and y–intercept.

(Note: You do not need to calculate uncertainties for Δm and Δb since these are uncertainties themselves!)

4.7.3 Large and Small Scatter and Instrument Precision

Since the size of the error bars will depend on the precision of the measuring instruments, whichever case you wind up with indicates whether there is any point in using more precise measuring instruments.

If you have the case of small scatter, then the uncertainty in the slope and y-intercept depend on the error bars. In this case, getting more precise measuring instruments would reduce the error bars and thus the uncertainties. If you have the case of large scatter, the uncertainties don't depend on the error bars, so more precise measurements wouldn't help.

4.7.4 Sample Least Squares Calculations

Following is a calculation of the least squares fit and the standard error of the slope and intercept for some test data.

$$N\left(\sum x_i^2\right) - \left(\sum x_i\right)^2 = (4)(0.3) - (1)^2 = 0.2$$

$$b = \frac{\left(\sum y_i\right)\left(\sum x_i^2\right) - \left(\sum x_i\right)\left(\sum x_i y_i\right)}{N\left(\sum x_i^2\right) - \left(\sum x_i\right)^2} = \frac{(16)(0.3) - (1)(4.3)}{0.2} = 2.5$$

$$m = \frac{N\left(\sum x_i y_i\right) - \left(\sum x_i\right)\left(\sum y_i\right)}{N\left(\sum x_i^2\right) - \left(\sum x_i\right)^2} = \frac{(4)(4.3) - (1)(16)}{0.2} = 6.0$$

$$S = \sum y_i^2 - m \sum x_i y_i - b \sum y_i = (66) - (6)(4.3) - (2.5)(16) = 0.2$$

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i	x_i	x_i^2	$x_i y_i$	y_i	y_i^2
1	0.1	0.01	0.3	3	9
2	0.2	0.04	0.8	4	16
3	0.3	0.09	1.2	4	16
4	0.4	0.16	2.0	5	25
N	$\sum x_i$	$\sum x_i^2$	$\sum x_i y_i$	$\sum y_i$	$\sum y_i^2$
4	1.0	0.3	4.3	16	66

Table 4.2: Sample Least Squares Fit Data

$$\sigma = \sqrt{\frac{S}{N-2}} = \sqrt{\frac{0.2}{4-2}} = 0.316228$$

$$\sigma_b = \sigma \sqrt{\frac{\sum x_i^2}{N\left(\sum x_i^2\right) - \left(\sum x_i\right)^2}} = (0.316228)\sqrt{\frac{0.3}{0.2}} = (0.3878298)$$

$$\sigma_m = \sigma \sqrt{\frac{N}{N\left(\sum x_i^2\right) - \left(\sum x_i\right)^2}} = (0.316228)\sqrt{\frac{4}{0.2}} = (1.414214)$$

Thus, if our data are such that σ_b and σ_m are the uncertainties in the y–intercept and the slope, and thus Δb and Δm, then

$$b = 2.5 \pm 0.4$$

and

$$m = 6 \pm 1$$

4.8   References

- *The Analysis of Physical Measurements*, Emerson M. Pugh and George H. Winslow, Addison-Wesley Series in Physics, 1966, QC39.P8

- *Errors of Observation and Their Treatment*, J. Topping, Chapman and Hall Science Paperbacks, 1972(4th Ed.)

- *Statistics*, Murray R. Speigel, Schaum's Outline Series in Mathematics, McGraw-Hill, 1961

Chapter 5

Standing Waves on a String

5.1 Purpose

In this experiment we will attempt to use the properties of standing waves on a string to measure the mass density of a string in order to determine which of two sample strings it matches.

5.2 Introduction

This experiment will introduce *graphical analysis*, including linearization, graphing and least squares fitting.

> This experiment illustrates the situation where you can't directly measure a quantity, but instead must use other physics principles to determine it indirectly. The *process* you go through will take several weeks, as in PC131. The physics involved will be covered in the first couple of labs, so most of your focus will be on how to analyze the results and prepare the report.

This lab will actually be broken into parts, so you will spend several weeks to produce the report. After you know how to do this, you will be able to produce reports much more quickly.

The schedule will be somewhat like this:

- Collect data, including uncertainties in the raw data.

- *Figure out how to produce a linear graph from the data.*

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- *Make a linear graph, and from the graph determine the slope and y-intercept and their uncertainties.*

- Using the graphical quantities, i.e the slope and y-intercept and their uncertainties, calculate numerical results, including uncertainties.

- Interpret the results and draw quantitative and qualitative conclusions.

- Learn how to write up "Discussion of Uncertainties" and "Conclusions" in your report.

(Notice that two of the steps are new this term; the rest are the same as in PC131.)

After that, you will hand in the lab.

5.3 Theory

Consider first a very long string, one end of which is firmly fixed in position (very far away), the other end of which is in your hand. If some tension is kept in the string, and you give the string a flick with your wrist, a pulse will originate from your hand and travel along the string away from your hand. The motion of your hand is perpendicular to the direction in which the pulse travels along the string or **transverse** to it. This transverse motion of the string results in each point on the string moving transverse to the motion of the wave as the wave passes.

Next consider the same long string. If the hand is moved quickly up and down in a regular manner, keeping the string taut, a wave will be generated which will travel along the string away from your hand. This is a **transverse wave** because each particle of the string moves up and down while the wave itself moves horizontally along the string. The wave itself is *traveling* along the string, and thus we have created a **traveling transverse wave** in the string.

This wave will appear, to a person viewing the string from the side, as alternating crests and troughs which move steadily away from the hand along the string. If this viewer watches one particular particle of the string, he will see it move up and down regularly, transverse to the motion of the wave.

Considering this traveling wave, if the viewer took a "fast" snapshot of it, it would look somewhat like Figure 5.1. (The dotted line represents where
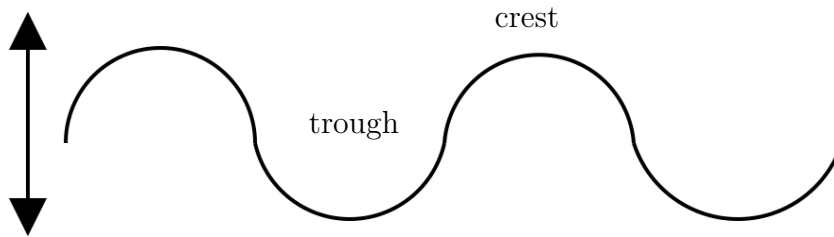
Figure 5.1: Creating a Wave "By Hand"
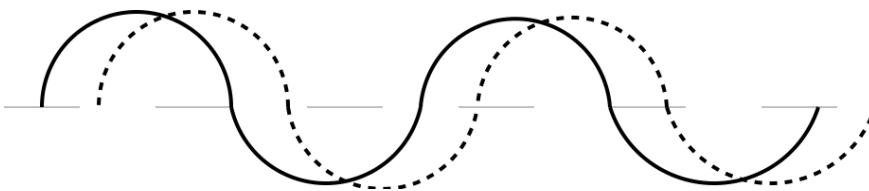
Figure 5.2: Traveling Wave at Different Times

the string would lie if the hand were at rest and the hand and string were both "frozen" for the duration of the snapshot.)

Neglecting the hand, if we were to take another snapshot shortly after the first and superimpose the two snaps, (drawing in the later one as a dashed line), we might see something like Figure 5.2, (i.e. the crests and troughs are moving steadily to the right along the string with a speed v. If we take a rapid succession of such snaps, we will soon find one in which the string lies exactly over the string in the first picture taken; each crest (or trough) has moved precisely to the position where there was a crest (or a trough) in the first snap.

We say that the distance this wave has traveled is one **wavelength**, λ. It is exactly equal to the distance between any consecutive pair of crests (or troughs) in one of the pictures. The wave thus moves one wavelength λ at speed v. The time for this to occur is called a period P of the wave. It is also the time taken for the hand at the end of the string to complete one of its regular (periodic) cycles of oscillation.

The wave's velocity, v, is thus equal to the distance λ it traveled in one

period P, i.e.

$$v = \frac{\lambda}{P} \tag{5.1}$$

In describing the hand's motion, which repeats itself every period P, it is perhaps more common to think of the number of times per second that the hand makes this cyclical motion. This quantity we call the **frequency**,f, of oscillation of the hand, and thus of the wave itself. If the length of time per cycle is the period P (in seconds), then the frequency f (in cycles/second, or **Hertz**) is its reciprocal.

$$f = \frac{1}{\frac{\text{time}}{\text{cycle}}} = \frac{\text{cycles}}{\text{time}} = \frac{1}{P} \tag{5.2}$$

(Note: 25Hz = 25 cycles per second, or cps.)

We now have two equations relating the wave's period, frequency, wavelength, and velocity; namely

$$v = \frac{\lambda}{P} \tag{5.3}$$

and

$$v = f\lambda \tag{5.4}$$

It can also be shown theoretically that the velocity of the wave traveling along the string is given by

$$v = \sqrt{\frac{T}{\mu}} \tag{5.5}$$

where T is the **tension** in the string and μ is the **mass** *per unit length* of the string, i.e. its **linear mass density**.

Now, consider the same string, but with much shorter length, so that the fixed end is now close to the free end. Also, consider that the free end is now being "shaken" by an electromechanical device instead of the hand of the now–exhausted student. The electromechanical device can shake the string periodically at a higher frequency than the hand for as long as necessary without fatigue, unlike the student.

The waves generated by the shaker travel along the string as previously, but they now strike the fixed end and are **reflected** there. The actual shape of the string at any instant is a combination of the incident and reflected

waves, and may be quite complex. However, under the proper set of conditions for T, v, λ, and f, the incident and reflected waves may combine producing regular shapes on the string. Such waves are called *stationary* or **standing waves**.

These standing waves occur if and only if the distance between the fixed and shaken ends of the string, (which is simply its length, L), is some integral number of *half-wavelengths* of the traveling wave. The waves then "fit" nicely into the length of the string. The incident and reflected waves reinforce each other over regions of large amplitude of vibration of the string called **loops**, and cancel each other at points on the string called **nodes** at which the string exhibits no motion at all. Examples of this are shown in Figure 5.3.

Our governing equation is a combination of Equations 5.4 and 5.5

$$f = \frac{1}{\lambda}\sqrt{\frac{T}{\mu}} \tag{5.6}$$

from which the mass density of the string may be determined. In this experiment, the oscillator frequency is fixed and known, and the string tension T is to be varied with resulting wavelengths λ measured for several different standing waves. The linear mass density μ will be determined, and also measured.

Equation 5.6 can be linearized so that a graph can be produced from which μ can be determined if f is known.

5.4 Procedure

5.4.1 Preparation

Since you may be doing this on your first week of labs, there are no pre-lab requirements.

5.4.2 Investigation

The apparatus is set up with a string of length L between fixed and oscillating ends.
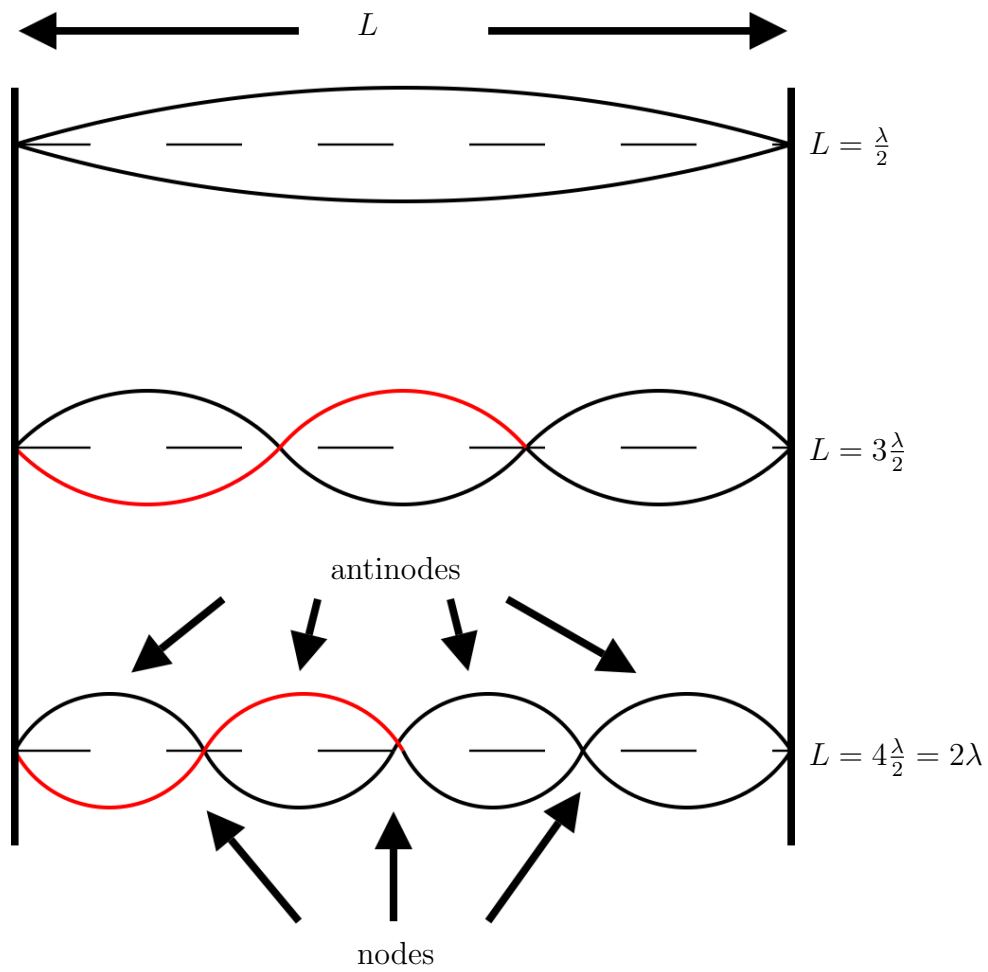
Figure 5.3: Standing Wave Patterns
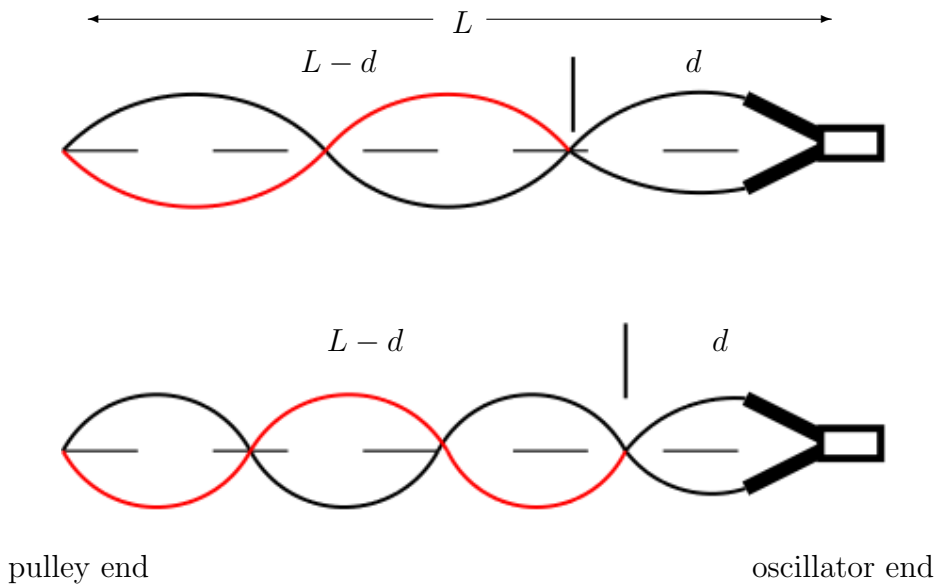
Figure 5.4: Measuring Wavelength

Method

1. Measure L and enter the result in Table 5.2. (Be sure to work in MKS units.) Determine the realistic uncertainty in L and fill in the details in Table 5.4.

2. Plug in the oscillator. The "fixed" end is where the string lies over the pulley. Tension in the string is produced by a weight consisting of a cup filled with a variable amount of copper shot. Adjusting the amount of shot in the cup adjusts the tension in the string.

3. Pour in shot a little at a time until a standing wave pattern with several, (preferably 7 or 8), loops is achieved. (Sometimes it helps to tug on the string by hand to determine whether weight needs to be added or subtracted, as well as to get an idea of how much weight to add or subtract.)

4. Once this is achieved, measure the distance, d, from the oscillator to the first node. *Note that you should not use the loop closest to the*

oscillator in your calculations as the "node" at one end is not a part of the string at all. Enter the data in Table 5.5.

5. Weigh the load pan with the shot in it, and enter the data in Table 5.5.

6. Adjust the mass until you see a noticeable change in the standing wave, (for instance, when it becomes unstable), and weigh the pan again. You can adjust the mass either up or down, whichever is easier. Record this "adjusted mass" in Table 5.5.

   Use the difference between the mass and the adjusted mass to determine the uncertainty in the mass, Δm.

7. Add shot to the cup until you get a standing wave pattern with one fewer loop than before and repeat the steps 4 to 6 above for each pattern up to and including one with just two loops.

8. If you can add enough mass to get a standing wave with a single loop, then do so and record the mass. (In this case, your calculation of the wavelength will be less precise due to the inclusion of the oscillator end, but there is no way to avoid it for this case.)

9. Calculate λ using the equations from **IQ7:**, and fill in Table 5.6. If the values are not changing monotonically, you need to make note of the following:

   You may find your data actually seems to be giving you two different frequencies of oscillation, where the higher frequency is twice the lower. This is due to the physical operation of the oscillator, and can be explained fairly simply, but that will not be done here. If this is the case for your data, calculate the "effective wavelength" for the odd points by multiplying or dividing the calculated wavelength by 2, (whichever is appropriate), and identify these points in your data and on your graph.[1]

---

[1]Normally it's not a good thing to "adjust" data like this. In this case, however, since the physics of the oscillator makes it oscillate at two frequencies which differ by an *exact* factor of 2, then it makes sense. Also, since the points being adjusted are clearly identified, it is not an attempt to deceive the reader.

10. The linear mass density of the string, μ, is measured by selecting a piece of the same kind of string, of length ℓ, and weighing it on a balance. Measure μ and ℓ for both sample strings and enter the results in Table 5.2.

11. Determine the realistic uncertainty in ℓ and fill in the details in Table 5.4.

12. It should be obvious that

$$\mu = \frac{\text{mass of string}}{\text{length of string}} = \frac{m_s}{\ell} \tag{5.7}$$

where m_s is the mass of the sample string from the balance. (Note that μ is not *volume* mass density, but *linear* mass density.) Determine the mass densities of the two sample strings with their uncertainties.

In-lab Questions

IQ1: Is there any reason that measuring between nodes should be better than measuring between antinodes? Explain.

IQ2: What is the uncertainty in the number of nodes, N? Explain.

IQ3: What is the realistic uncertainty in the length used for each measurement? Is it more due to getting the right mass to make the pattern stable, or is it more due to measuring the position of the node? Does this vary during the experiment? Explain.

IQ4: What is the realistic uncertainty in the mass used for each measurement? Does the shot size affect this? Explain.

IQ5: What is the realistic uncertainty in the length used for the mass density calculation, and what causes it?

IQ6: Why should using a long piece of string be better for calculating the mass density than a short one?

IQ7: If the total length of the string is L, and the last node (near the oscillator) has a length d, as in Figure 5.5 then determine the equation for the wavelength λ if there are N nodes visible.
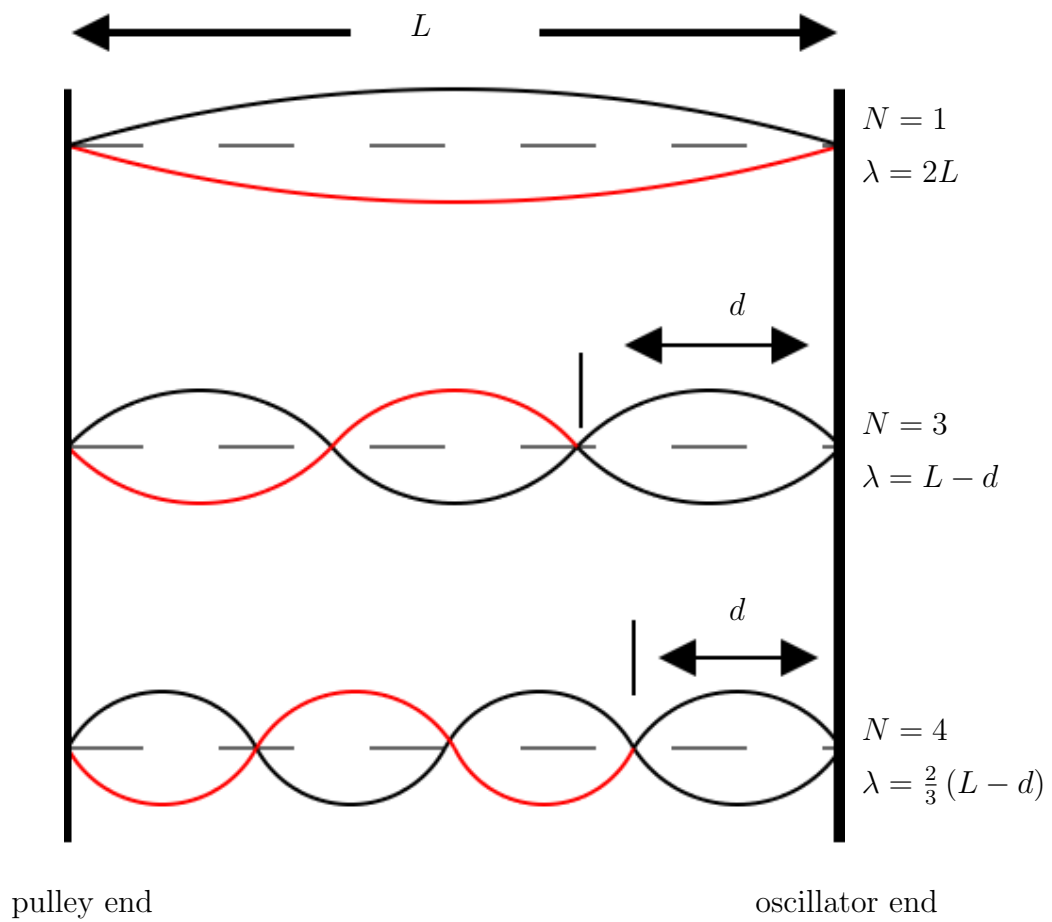
Figure 5.5: Wavelength Relation to d

For instance,

- for $N = 1$, $\lambda = 2L$.

- for $N = 2$, $\lambda = 2(L - d)$.

- for $N = 3$, $\lambda = (L - d)$.

- for $N = 4$, $\lambda = \frac{2}{3}(L - d)$.

- for $N = 5$, $\lambda = \frac{2}{4}(L - d)$.

- $\cdots$

(It might help to note the pattern in the last two, and then re-write the previous two in the same form to determine the general equation. *Note the first one does not fit the equation, since it uses the node ending at the oscillator and the others don't.*)

In-lab Tasks

IT1: Fill in the first and third columns of Table 5.6 to see that both are increasing monotonically.

IT2: Rearrange Equation 5.6 to solve for μ. Do an order of magnitude calculation[2] of the linear mass density of the string using one data point to show that it is in the right range.

IT3: Read over each of the inlab and postlab questions, and decide where the answers should appear in your lab report. (Note that some questions may have parts of the answers in each section.) Fill in the results in Table 5.7.

5.4.3 Analysis

This lab can't be completed until after the *"Exercise on Graphing and Least Squares Fitting using a Spreadsheet"* lab exercise.

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[2]If you use a single data point to quickly calculate something, rather than using all of the points for an average, graph, etc., it serves the same purpose as an order of magnitude calculation; it lets you see quickly if the results seem to make sense.

1. For each of the three linearizations of your data, plot the graph, perform the least squares fit, and calculate the linear mass density, μ, and its uncertainty from the results.

2. Using the equations for linear mass density, μ, and its uncertainty from the linearizing exercise, determine linear mass density and its uncertainty from each linearization.

3. Consider the similarities and differences in results of the three linearizations, and decide which one you will use for your lab report. *If there are big differences between the values of R^2 for the different linearizations, you should see a difference in how linear each graph appears to be. Different types of errors in the original data will affect the different linearizations differently, and so if there's a big difference you will have a hint at what errors may exist in your data.*

4. Depending on the linearization chosen, use the slope or y-intercept as appropriate to determine a value for μ, the linear mass density of the string.

5. Compare the value of μ obtained to the values obtained from the string samples.

6. Determine whether the string matches either of the samples, and if so, which one.

Post-lab Discussion Questions

Answers to the following questions will form the basis of the *Discussion* and *Conclusions* sections of your lab report. Write these sections in paragraph form, with each individual answer underlined or highlighted. At the beginning of each question put the question number in super-script. The goal is to have a flow to the whole section, rather than to have the section appear as a series of statements of unrelated facts. *Be sure to include your numerical results to explain your answers.*

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Comparing linearizations

Q1: Did the linear mass densities given by the different linearizations agree within experimental uncertainty? Discuss the similarities and differences in results of the three linearizations, and explain which one you prefer and why. Did your original preference from *"Exercise on Linearizing Equations"* still make sense after you had plotted the data and done the calculations? Why or why not?

For your preferred linearization

Q2: For your linearization, one of the graphical parameters (either the slope or the y-intercept), was used to determine the linear mass density. The other should have equaled a constant. Did the other parameter give you the expected value for the constant? Explain.

Q3: From the shape of your linearized graph, is there any evidence of a systematic error in your data? Explain.

Experimental insight

Q4: Based on whether the scatter in your graph was large or small, would more precise measuring instruments have improved your results? Explain.

Q5: Based on your answer to **Q4**, would a more precise metre stick have improved your results? Explain. (Your inlab questions should help you answer this.)

Q6: Based on your answer to **Q4**, would a more precise balance have improved your results? Explain. (Your inlab questions should help you answer this.)

Q7: Does the magnitude of your calculated mass density agree with either of the sample strings in the lab? Can you identify which type of string you had?

5.5 Bonus

What factor determines the maximum number of nodes which can be observed? Are data more reliable for oscillations with many nodes or with few

nodes? Explain your answers.

5.6 Recap

By the time you have finished this lab report, you should know how to :

- collect data and analyze it, which may include

    - graphing with error bars
    - least squares fitting

- write a lab report which includes:

    - *title* which describes the experiment
    - *purpose* which explains the objective(s) of the experiment
    - *results* obtained, including data analysis
    - *discussion of uncertainties* explaining significant sources of uncertainty and suggesting possible improvements
    - *conclusions* about the experiment, which should address the original objective(s).

5.7 Summary

Item	Number	Received	weight (%)
Pre-lab Questions	0	_____	0
In-lab Questions	7	_____	70
Post-lab Questions	7	_____	*in report*
Pre-lab Tasks	0	_____	0
In-lab Tasks	3	_____	30
Post-lab Tasks	0	_____	0

5.8   Template

My name:
My partner's name:
My other partner's name:
My lab section:
My lab demonstrator:
Today's date:
Your string identifier is:

quantity	symbol		single/
		given/ mine	repeated/ constant
Not in equations			

Table 5.1: List of quantities

symbol	value	units	instrument			effective uncertainty
			reference (e.g. A.1)	precision measure	zero error	
L						
Not in equations						

Table 5.2: Single value quantities

quantity	symbol	equation	uncertainty
	T		
	λ		
	μ		

Table 5.3: Calculated quantities

symbol	factor		bound	units
	Sources of *systematic* error			
	Sources of *random* error			

Table 5.4: Experimental factors responsible for effective uncertainties

Instrument						
reference (or name)						
units						
precision measure						
zero error						
number of 1/2 wavelengths		distance from end	uncertainty in distance	hanging mass	adjusted mass	uncertainty in mass
total N	used	d	Δd	m	$m \pm \Delta m$	Δm
		( )			( )	
1	1*	0				
2	1					
3	2					
4	3					
5	4					
6	5					
7	6					

Table 5.5: Resonance raw data

λ	$\Delta\lambda$	T	ΔT
( )	( )	( )	( )

Table 5.6: Modified data

Where to answer		
Question number	Discussion (y/n)	Conclusions (y/n)
In-lab		
Post-lab		
Hints		
	"think"	"agree"
	"suggest"	"equal"
	"explain"	"do (did, does) "
	"how"	"significantly different"
	"why"	"support"
	"what"	"verify"

Table 5.7: Lab Report Organization

Chapter 6

Exercise on Linearizing Equations

6.1 Purpose

The purpose of the exercise is to develop skills in producing linear graphs from various types of data and extracting results.

6.2 Introduction

This exercise will develop skills in linearizing data, so that a variety of relationships can be graphed as straight lines.

6.3 Theory

Often, the point of a scientific experiment is to try and find empirical values for one or more physical quantities, given measurements of some other quantities and some mathematical relationship between them. For instance, given a marble has a mass of 5 g, and a radius of 0.7 cm, the density of the marble can be calculated given that $v = 4/3\pi r^3$ and $\rho = m/v$. (For the sake of simplicity, uncertainties will be ignored for now, although the calculation of those should be familiar by now.)

Many times, however, rather than having *one* measurement of a quantity, or set of quantities, we may have several measurements which should
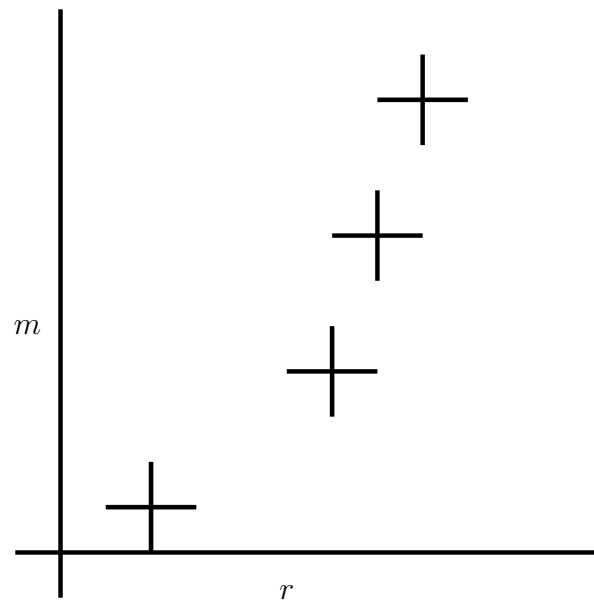
Figure 6.1: Non-linear equation

all follow the same relationships, (such as if we had several marbles made of the same material in the example above), and we wish to combine the results. The usual way of combining results is to create a graph, and extract information (such as the density) from the slope and y–intercept of the graph.

One may be tempted to ask why a graph should be better than merely averaging all of the data points. The answer is that an average is completely unbiased. The variation of any one point from the norm is no more or less important than the variation of any other point. A graph, however, will show any point which differs significantly from the general trend. Analysis of the graphical data (such as with a least squares fit) will allow such "outliers" to be given either more or less weight than the rest of the data as the researcher deems appropriate. Depending on the situation, the researcher may wish to verify any odd point(s), or perhaps the trend will indicate that a linear model is insufficient. In any case, it is this added interpretive value that a graph has which makes it preferable.

> *A plot is better than an average since it may indicate systematic errors in the data.*

The value in fitting the data to an equation is that once the fit has been done, rather than continuing to work with a large amount of data, we can simply work with the parameters of our fit and their uncertainties. In the case of a straight line, all of our data can be replaced by four quantities; $m, \Delta m, b$ and Δb.

> *A fit equation replaces a bunch of data with a few parameters.*

The reason a linear graph is so useful is that it's easier to identify whether a line is straight than it is to identify whether it looks more like $y = x^2$ or $y = x^3$, for instance.

> *A straight line is easy to spot with the unaided eye.*

If the data fits an equation of the form $y = mx + b$, then it is easy to plot a straight line graph and interpret the slope and y–intercept, but it is rarely that simple. In most cases, the equation must be modified or **linearized** so that the variables *plotted* are different than the variables *measured* but produce a straight line.

Linearizing equations is this process of modifying an equation to produce new variables which can be plotted to produce a straight line graph. In many of your labs, this has been done already.

Look again at $y = mx + b$. Note that y and x are *variables*, (as each can take on a range of values), while m and b are *constants*, (as there is only one value for each for all of the data points). We can linearize an equation if we can get it in the form

$$variable_1 = constant_1 \times variable_2 + constant_2$$

There are a few things to note:

1. Several constants combined together produces another single constant.

2. Powers or functions of constants are also constants.

3. Constants may have "special" values of 0 or 1 so they appear "invisible". For example
$$y = mx$$
   is still the equation of a straight line, where $b = 0$. As well,

$$y = b$$

   is the equation of a line where $m = 0$.

4. Variables may be combined together to form new variables.

5. Powers or functions of variables are also variables.

> *Note that linearizing an equation will produce expressions for the slope and y–intercept which depend only on the constants in the original equation,* not *on the original x and y variables. This means that the constants can be related to the slope and y–intercept rather than the original variables.*

6.3.1 Techniques for Linearization

If a relationship involves *only* multiplication and division, (including powers), then logarithms can be used to linearize. Sometimes taking roots or powers of both sides of an equation will help.

6.3.2 Procedure for Linearization

The steps are as follows:

1. Rearrange the equation to get one variable (or a function of it) on the left side of the equation; this becomes your y variable.

2. Regroup the right side of the equation to create a term containing the other variable (or some function of it).

3. Use the left-side variable (or the function of it) as your x variable, and then your slope should be whatever multiplies it; your y intercept is whatever additive term is left over.

Note: It is important to realize that you don't need to understand an equation to linearize it; all you have to know is which parameters are variables (i.e. things you have data for), and which parameters are constants (i.e. things you want to calculate). Of course different experiments involving the same relationship may make different parameters variable, and so how an equation is linearized will depend on the data used. To again consider the above example: The original equations were

$$v = (4/3)\pi r^3 \tag{6.1}$$

and

$$\rho = m/v \tag{6.2}$$

where the quantities m and r are measured. (i.e. We have several marbles of the same material, so we can get several measurements of m and r, but we expect ρ to be the same for all of them.) Thus for this situation, m and r are variables, and ρ is a constant. We can combine the two equations to get

$$\rho = \frac{m}{(4/3)\pi r^3} \tag{6.3}$$

or

$$\rho = \frac{3m}{4\pi r^3} \tag{6.4}$$

This equation has a constant on one side, and a mixture of variables and constants on the other. First we should rearrange it to get a variable on the left hand side. Suppose we rearrange the equation, giving

$$m = (4/3)\pi\rho r^3 \tag{6.5}$$

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This leaves a variable on the left. From this point on, there are two main possibilities for how to proceed: [1]

Method I

Now we can create a new variable, Y such that

$$Y = m$$

By the rule about powers of variables being variables, then we can create a new variable X given by

$$X = r^3$$

Then equation 6.5 above becomes

$$Y = (4/3)\pi\rho X \tag{6.6}$$

since π is a constant, and ρ *should* be, and using the rule that combinations of constants produce constants, then we can define M, a constant, (not the same as m), as

$$M = (4/3)\pi\rho$$

so equation 6.6 becomes

$$Y = MX + 0$$

which is the equation of a straight line. (In the case, B, the y–intercept is zero.)[2] So if we plot our "modified" variables, we should get a straight line, passing through the origin with a slope M. How can we get ρ from the graph? Well, from above

$$M = (4/3)\pi\rho$$

so

$$\rho = \frac{3M}{4\pi}$$

where M is the slope of the graph.

---

[1]Usually the process is not as explicit as this. i.e. one doesn't usually create an X and a Y, but doing this illustrates the procedure.

[2]Occasionally we can get a situation where the slope is similarly "invisible", if it is 1 or 0.
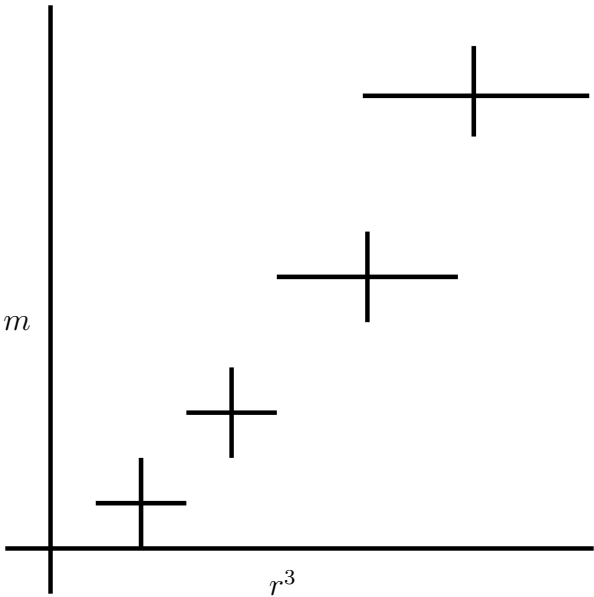
Figure 6.2: One linearization

Method II

We can take logarithms of both sides, so that Y such that equation 6.5 above becomes

$$\ln m = \ln\left((4/3)\pi\rho\right) + \ln r^3 \tag{6.7}$$

grouping the terms so one only contains constants (and so the combination should be constant) and one only contains the variable r. We can bring down the exponent so equation 6.7 becomes

$$\ln m = \ln\left((4/3)\pi\rho\right) + 3\ln r$$

Now we can create new variables, Y such that

$$Y = \ln m$$

and

$$X = \ln r$$

which is the equation of a straight line. So if we plot our "modified" variables, we should get a straight line. How can we get ρ from the graph? Well, from above

$$B = \ln\left((4/3)\pi\rho\right)$$

so

$$\rho = \frac{3}{4\pi}e^B$$

where B is the y-intercept of the graph. (In this case, the value you get from the graph for the slope should suggest whether the fit is a good one.)

> *Remember that after linearization, our results depend on our graphical quantities of the slope and the y-intercept, rather than on the original measured quantities.*

6.3.3   Choosing a Particular Linearization

Often there may be more than one linear form for the equation so there may be more than one "right answer". In this case, there are a few things which may help you choose.
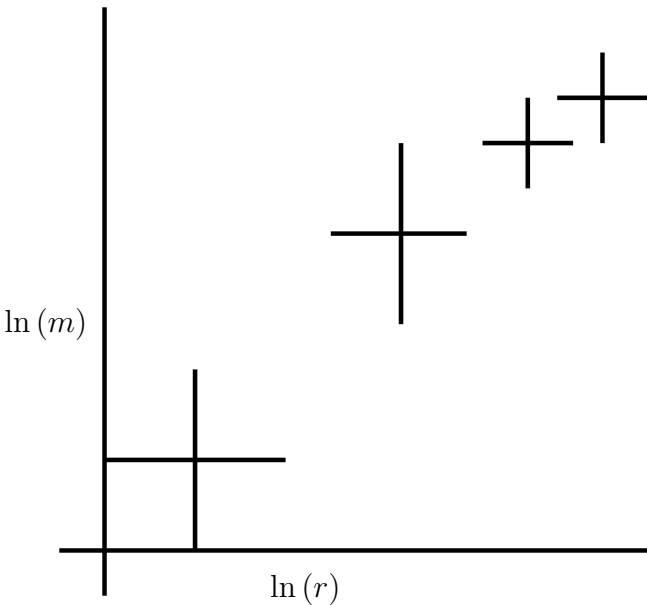
Figure 6.3: Another linearization

Simple variables

A preferable linearization is one which most simplifies understanding the graph or interpreting the results. For instance, in the above example, it would have been possible to use $(4/3)\pi r^3$ instead of r^3 as our x variable, but that would make confusing axis scales and/or units (although it would have made the slope be ρ with no calculation).

Spread of data

The spread of data will be different for each linearization. A graph with points which are more equally spaced is generally preferable to one where the points are concentrated in one area.

Size of error bars

Like the spread of data, the size of the error bars will be different for each linearization. A graph with more equally sized error bars is generally preferable to one where the error bars vary greatly in size for different points.

> *Usually it is preferable to separate variables and constants as much as possible in your linearization so that graph variables are easily related to experimental ones.*

6.3.4 Uncertainties in Results

After determining how equation parameters relate to graphical quantities as above, uncertainties can be determined as usual. In the above example Method I gives

$$\Delta\rho = \frac{3\Delta M}{4\pi}$$

while for Method II

$$\Delta\rho = \frac{3}{4\pi}e^B \Delta B$$

or

$$\Delta\rho = \rho\Delta B$$

6.3.5   Systematic Error in Variables

If there is systematic error in your data, it will show up on the graph.

- If your slope or y-intercept should have a specific value, such as zero, then a value other that that suggests systematic error.

- Since your graph is supposed to be linear, then if the points actually seem to be on a curve rather than a line there is probably systematic error in the data

If systematic error is suggested in either of these ways, you should try to determine which variable(s) may be responsible, and whether your data values are above or below their correct value.

6.4   Procedure

6.4.1   Preparation

You are welcome to go ahead and do as much of the exercise on your own as you wish; you can just bring a sheet with your question answers to the lab. If you get it all done in advance, that's great.

6.4.2   Investigation

The fundamental frequency of vibration of a string is given by

$$f = \frac{1}{\lambda}\sqrt{\frac{T}{\mu}}$$

λ and T, are measured variables and f is a constant, with an uncertainty Δf. Determine μ and $\Delta \mu$.

There are 3 different ways of doing this. In each case, show what should be plotted, how the error bars for your x (independent) and y (dependent) variables are determined, what the slope and y-intercept will be, and how μ and $\Delta \mu$ come from the slope and y-intercept and their uncertainties.

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For instance, from the example above, using Method I we would say:

- Plot m vs. r^3. (In other words, the independent (x) variable is r^3 and dependent (y) variable is m.)

- The uncertainty in the dependent variable is Δm.

- The uncertainty in the independent variable is $3r^2\Delta r$.

- The slope of the graph will be $M = (4/3)\pi\rho$.

- The y-intercept should be zero[3].

- The density will be determined from the slope by the equation $\rho = \frac{3M}{4\pi}$.

- The uncertainty in the density will be determined from the slope by the equation $\Delta\rho = \frac{3\Delta M}{4\pi}$.

In-lab Questions

For each of the linearizations below, be sure to address all 7 points as in the example above.

IQ0: What is the linearization if our dependent variable is λ? *(See the example above.)*
Answer: First we have to rearrange the equation

$$ f = \frac{1}{\lambda}\sqrt{\frac{T}{\mu}} $$

to get the *dependent* variable on the left, as follows:

$$ \lambda = \frac{1}{f}\sqrt{\frac{T}{\mu}} $$

Then we need to regroup the right side to separate the *independent* variable[4].

$$ \lambda = \frac{1}{f\sqrt{\mu}}\sqrt{T} $$

---

[3]If the y-intercept turns out to be something other than zero, then there is some systematic error in our experiment.
[4]There is another possibility here; see if you can figure out what it is.

So the slope is the part that is multiplied by the independent variable,

$$\textbf{slope} = \frac{1}{f\sqrt{\mu}}$$

which means

$$\sqrt{\mu} = \frac{1}{\textbf{slope}f}$$

and so

$$\mu = \frac{1}{\textbf{slope}^2 f^2}$$

The uncertainties in each of the quantities are calculated as usual. Specifically, if μ and f have uncertainties $\Delta\mu$ and Δf, and after the least squares fit, the slope and y-intercept will simply be quantities m and b with associated uncertainties Δm and Δb, then

$$\Delta\mu = \mu\left(\frac{\Delta\textbf{slope}^2}{\textbf{slope}^2} + \frac{\Delta f^2}{f^2}\right)$$
$$= \mu\left(\frac{2\textbf{slope}\Delta\textbf{slope}}{\textbf{slope}^2} + \frac{2f\Delta f}{f^2}\right)$$
$$= \mu\left(2\frac{\Delta\textbf{slope}}{\textbf{slope}} + 2\frac{\Delta f}{f}\right)$$
$$= 2\mu\left(\frac{\Delta\textbf{slope}}{\textbf{slope}} + \frac{\Delta f}{f}\right)$$

using algebra, and *given that the slope should be positive.* (The uncertainty determined by inspection is shown in the template in the example. Remember that it doesn't matter which method you use since they should usually give similar results to one significant figure.)

IQ1: What is the linearization if our dependent variable is λ^2? *(See the example above.)*

IQ2: What is the linearization if our dependent variable is $\ln(\lambda)$? *(See the example above.)*

IQ3: *Without considering the data*, which of the above choices would you prefer and why? (In other words, is there a choice that seems better to work with, regardless of how the data fit?)

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In-lab Tasks

IT1: Fill in the headings for the columns in Tables 6.1,6.2, and 6.3.

IT2: Fill in the template with the equations, uncertainties, and units for the slope and y-intercept for each linearization.

6.4.3 Analysis

Post-lab Tasks

There are no post-lab tasks for this exercise, although the results of this exercise will be used in later exercises.

Post-lab Discussion Questions

There are no post-lab questions for this exercise, although much of the analysis from here will belong in your lab report

6.5 Bonus

You can do one of the questions below. In each of the following questions, state the modified variables to be plotted, and state how the unknown(s) may be determined from the graph.

1. The position of a body starting from rest and subject to uniform acceleration is described by

$$s = \frac{1}{2}at^2$$

 s and t are measured variables. Determine a and Δa.

2. The period of oscillation of a simple pendulum is

$$\mathrm{T} = 2\pi\sqrt{\frac{L}{g}}$$

 where L and T are measured variables. Determine g and Δg.

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3. The Doppler shift of frequency for a moving source is given by

$$f = f_0 \frac{v}{v - v_0}$$

f and v_0 are measured variables, f_0 is fixed and known. Determine v and Δv. (*Note that even though the notation might suggest otherwise, in this case v_0 is a variable, not a constant.*)

4. The impedance of a series RC circuit is

$$Z = \sqrt{R^2 + \frac{1}{\omega^2 C^2}}$$

Z and ω are measured variables. Determine R , ΔR, C, and ΔC.

5. The conductivity of an intrinsic semiconductor is given by

$$\sigma = C e^{\frac{-E_g}{2kT}}$$

σ and T are measured variables and k is a known constant. Determine E_g, ΔE_g, C, and ΔC.

6. The relativistic variation of mass with velocity is

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

m and v are measured variables. Determine m_0, Δm_0, c, and Δc.

7. The refraction equation is

$$\mu_1 \sin \theta_1 = \mu_2 \sin \theta_2$$

θ_1 and θ_2 are measured variables; μ_1 is fixed and known. Determine μ_2 and $\Delta \mu_2$.

6.6   Recap

By the end of this exercise, you should understand the following terms:

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- linear graph

- linearized equation

You should also be able to:

- identify combinations of variables which should result in a linear graph

- produce a linear graph from any linearizable equation

6.7   Summary

Item	Number	Received	weight (%)
Pre-lab Questions	0	_____	0
In-lab Questions	3	_____	75
Post-lab Questions	0	_____	0
Pre-lab Tasks	0	_____	0
In-lab Tasks	2	_____	25
Post-lab Tasks	0	_____	0
Bonus		_____	5

6.8   Template

Trial number	independent $\sqrt{T}$	dependent λ
1		
2		
3		
4		
5		

Table 6.1: λ Linearization

For this linearization,
The equation for the uncertainty in the independent variable is:
$\Delta x = \Delta\sqrt{T} \approx \frac{1}{2\sqrt{T}}\Delta T$ (using algebra)
The equation for the uncertainty in the dependent variable is:
$\Delta y = \Delta\lambda$

The equation for the slope is: $\mathbf{slope} = \frac{1}{f\sqrt{\mu}}$
The equation for the y-intercept is: $\mathbf{y - intercept = 0}$

The units for the slope are: $\mathbf{mN^{-1/2}}$
The units for the y-intercept are: $\mathbf{m}$

The equation for the linear mass density, μ, is: $\boldsymbol{\mu} = \frac{1}{\mathbf{slope}^2 f^2}$
The equation for the uncertainty in the mass density, $\Delta\mu$, is:
$\Delta\mu = \Delta\left(\frac{1}{\mathbf{slope}^2 f^2}\right) \approx \frac{1}{\mathbf{slope}^2 f^2} - \frac{1}{(\mathbf{slope}+\Delta\mathbf{slope})^2(f+\Delta f)^2}$
(using inspection, given that the slope and f should be positive)

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Trial number	independent	dependent λ^2
1		
2		
3		
4		
5		

Table 6.2: λ^2 Linearization

For this linearization,
The equation for the uncertainty in the independent variable is:
The equation for the uncertainty in the dependent variable is:

The equation for the slope is:
The equation for the y-intercept is:

The units for the slope are:
The units for the y-intercept are:

The equation for the linear mass density, μ, is:
The equation for the uncertainty in the mass density, $\Delta\mu$, is:

Trial number	independent	dependent $\ln(\lambda)$
1		
2		
3		
4		
5		

Table 6.3: $\ln(\lambda)$ Linearization

For this linearization,
The equation for the uncertainty in the independent variable is:
The equation for the uncertainty in the dependent variable is:

The equation for the slope is:
The equation for the y-intercept is:

The units for the slope are:
The units for the y-intercept are:

The equation for the linear mass density, μ, is:
The equation for the uncertainty in the mass density, $\Delta\mu$, is:

Chapter 7

Exercise on Graphing and Least Squares Fitting using a Spreadsheet

7.1   Purpose

The purpose of this experiment is to become familiar with using spreadsheet to produce graphs and analyze graphical data.

7.2   Introduction

While a spreadsheet has a lot of useful features for data analysis, etc. it was designed primarily for use by people in business rather than for scientists, and so there are some things which scientists wish to do which require a little effort in a spreadsheet. Graphical analysis is one of the areas where this is true, as you shall see. (Even with the extra work, it's a lot more convenient than doing it all by hand.)

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7.3   Theory

7.3.1   Graphing

Graph Type

The graph type mainly used by scientists is an **x-y** (or *scatter*) graph[1]. *Do not* choose a line graph!

Colour

The default background colour, (grey in many cases), in many spreadsheets just looks bad in graphs; it obscures the data and serves no purpose. *Turn the background colour off!*

Gridlines

Grid lines should be either removed or in both dimensions. Gridlines in one direction only look odd on an "xy" graph. *Turn the gridlines off!*

Text

The main text of a graph consists of $x-$ and $y-$ titles, a *main title* and perhaps a *sub–title*. All of these may be set in a spreadsheet.

Series

A spreadsheet allows you to plot several different "series" of (x, y) data. Each series can be *customized*, with choices for many things, including the following:

- Patterns

  Each series can be plotted with lines, symbols, or both.

  > *Do not connect the points like a dot-to-dot drawing!*

---

[1] As long as there is some *mathematical relationship* between the variables, then an x-y graph illustrates the relationship. However, if the independent variable does not have a numerical value, then this doesn't apply. For instance, if you were graphing reaction time for men and women, then a bar graph would be the logical choice, since there's no *numerical* relationship between "men" and "women".

Do not use an arbitrary function just because it goes through all the data points!

- Markers

  There are many possible symbols which can be used for each series.

- Lines

  There are several line types available for each series.One important fact about how lines are used to connect points in a series; *all points in a series are joined by lines*, unless the line for that series is turned off.

In science, it is almost always wrong to have a dot-to-dot drawing. It is also wrong to have a curve which has no mathematical significance. For this reason, data points should not be connected by either line segments or a curve like a polynomial which is made to pass through each data point. The only line or curve which should be shown is the result of a fit which is based on some theoretical mathematical relationship.

Matching up x and y Values

When you create an xy graph is created in a spreadsheet, you *don't* input data values as (x, y) pairs. Instead you select series for each of x and y. The way the individual x and y values are associated is by where they occur in their respective series.

In Figure 7.1 you can see that the 5th point in each series is highlighted. Even though the series all start in different rows and columns, since the number of cells in each is the same, corresponding values can be considered to be related. (If a cell is blank, then the corresponding point or error bar will not be plotted.)

Error Bars

In a spreadsheet, when you choose *custom* error bars, you can choose series for both x and y, and even potentially different series for the $+$ and $-$ directions.
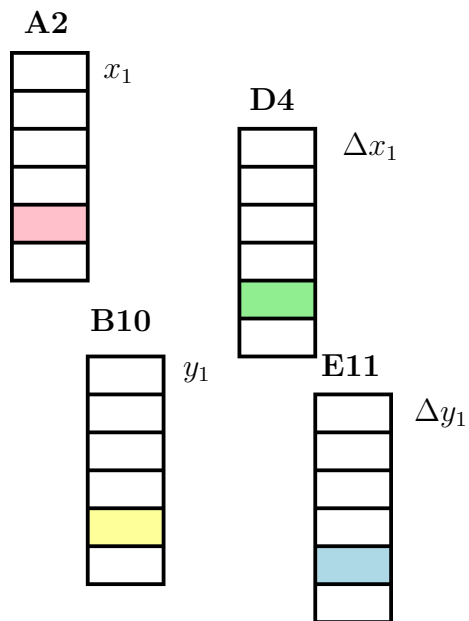
Figure 7.1: Spreadsheet layout with series for x, y, and error bars

You don't necessarily have to put markers on the ends of the error bars; the value in doing so is to make it clear that you're not just using "+" symbols for plotting the data points. Also, if you are including a grid on your graph, error bars without markers on the end may be hard to distinguish. However if the error bars will be clearly identifiable without markers, you don't need to use them.

7.3.2 Least Squares Fitting

The point of plotting a graph in an experiment is usually to extract information from the graph; often the data is plotted in such a manner that the model being tested suggests that the data should fit a straight line. If it does, then getting the slope and y intercept of the line of best fit along with their associated uncertainties is necessary. One of the two usual ways to determine the uncertainty in a graphical quantity is to calculate the **standard error**. (The other involves finding lines of maximum and minimum slope.) The following sections discuss using a spreadsheet to do least squares fitting and to calculate standard errors.

Determining the equation of the line by formulas

In Chapter 4, *"Graphs and Graphical Analysis"*, the lab manual explains how to calculate a least squares fit to a set of data. This can be done in a spreadsheet by creating additional cells corresponding to each data point which contain, respectively, x^2, y^2 and xy. At the end of the data, these quantities can be totaled to give the sums necessary to do the least squares fit.[2]

---

[2]You may notice that a particular quantity comes up a lot. It is

$$N \left(\sum x_i^2 \right) - \left(\sum x_i \right)^2 \tag{7.1}$$

It only takes a couple of lines of algebra to show that this equals

$$N \left(N - 1 \right) {\sigma_x}^2 \tag{7.2}$$

where ${\sigma_x}^2$ is the sample standard deviation of the x values.

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Determining the equation of the line using LINEST()

Excel and LibreOffice each contain a function to do least squares fitting. Unfortunately if produces a bunch of numbers without indicating what is what. It also has to be configured to do things the way we want. Make sure to configure it to give extra statistics.

> *When using **LINEST()** to calculate the least squares fit, always set it to calculate the y-intercept, even when you expect it to be zero! This gives you important information about the data.*

Comparing the result given by the least squares fit using your formulas with your regression output should indicate what several of the quantities are.

(If you use **LINEST()** to do least squares fitting for a lab report, quote the quantities given with the names used in the lab manual. The unidentified block of cells given by many spreadsheets is not very meaningful.)

Determining uncertainties in the slope and y-intercept

Case I: Maximum and minimum slopes   If the error bars are large enough, then the line of best fit should go through all of the error bars. In this case, there will be two data points which determine coordinates for a line of maximum slope which crosses all of the error bars. Consider the case for positive slope:

If we label two points x_1 and x_2, where $x_1 < x_2$, then we can see from Figure 7.2 that the steepest line which touches the error bars for both x_1 and x_2 is the line between $(x_1 + \Delta x_1, y_1 - \Delta y_1)$ and $(x_2 - \Delta x_2, y_2 + \Delta y_2)$. The slope of this line will then be
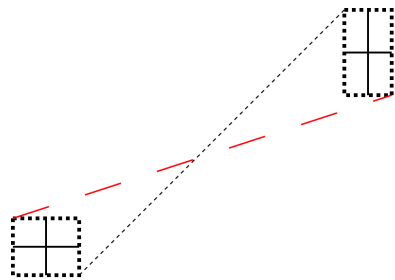
$$m_{max} = \frac{(y_2 + \Delta y_2) - (y_1 - \Delta y_1)}{(x_2 - \Delta x_2) - (x_1 + \Delta x_1)}$$
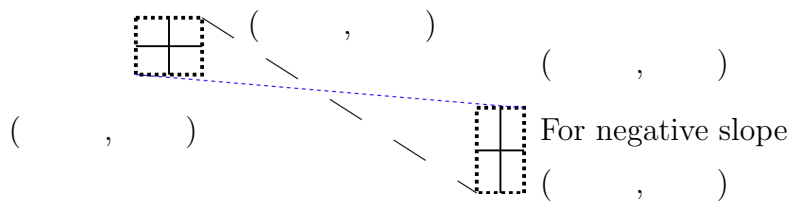
and then the y-intercept is given by

$$b_{min} = (y_1 - \Delta y_1) - m_{max}(x_1 + \Delta x_1) = (y_2 + \Delta y_2) - m_{max}(x_2 - \Delta x_2)$$

Similarly the line with the least slope which touches the error bars for both x_1 and x_2 is the line between $(x_1 - \Delta x_1, y_1 + \Delta y_1)$ and $(x_2 + \Delta x_2, y_2 - \Delta y_2)$. The slope of this line will then be

$$m_{min} = \frac{(y_2 - \Delta y_2) - (y_1 + \Delta y_1)}{(x_2 + \Delta x_2) - (x_1 - \Delta x_1)}$$

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For positive slope

(,)

(,)

(,)

For negative slope

(,)

Figure 7.2: Maximum and Minimum Slope Coordinates from a Point

and then the y-intercept is given by

$$b_{max} = (y_1 + \Delta y_1) - m_{min}(x_1 - \Delta x_1) = (y_2 - \Delta y_2) - m_{min}(x_2 + \Delta x_2)$$

The case for a negative slope is shown in Figure 7.2; the analysis is left to the student.

> *The points for the maximum and minimum slope will not always be the endpoints on the graph.*

Index function   To calculate the slope and y-intercept from a block of data, we can use the *index* function. Its syntax is as follows:

- index(reference, row number, column number)

- reference is the cell range to look in

- row number (starts at one)

- column number (starts at one)

So if we have a data set of 6 values where the x values start in $A2$, and the Δx values start in $D4$, then we can get

$$x_2 + \Delta x_2$$

by the formula

$$= \textbf{INDEX}(\textbf{A2} : \textbf{A7}, \textbf{2}, \textbf{1}) + \textbf{INDEX}(\textbf{D4} : \textbf{D9}, \textbf{2}, \textbf{1})$$

(Note that the only difference is in which block of data to use.)   You'd probably write the formula as

$$= \textbf{INDEX}(\$\textbf{A}\$\textbf{2} : \$\textbf{A}\$\textbf{7}, \textbf{2}, \textbf{1}) + \textbf{INDEX}(\$\textbf{D}\$\textbf{4} : \$\textbf{D}\$\textbf{9}, \textbf{2}, \textbf{1})$$

so that you could copy it and still refer to the same blocks of data.

Case II: standard errors   If the error bars are small enough, then the points will be scattered in such a way that no line can be drawn which crosses all of the error bars. In this case, the uncertainties in the slope and y-intercept reflect the scatter of the points. In this case, the uncertainty in the slope and y-intercept will be calculated using the **standard errors** in the slope and y-intercept, in much the same way that the uncertainty for an average value is calculated using the *standard error of the mean*.

7.3.3 Displaying Lines

Unless you are going to give the equation of a line or curve, do not *show it on a graph!*

Plotting arbitrary lines

To display a line on the graph, such as a best fit line, one can use a series which has not yet been used. When one knows the *equation* of a line, all one needs is two endpoints so that a line can be drawn between them. To allow this, include 2 values at the end of your x series, x_{min} and x_{max} which are the minimum and maximum values from the x data, respectively. Placing the y values calculated from the line equation in the corresponding cells of another series will allow a line to be plotted between those points. (Set the format for that series to lines only.)

Using "trendline"

There is a built-in feature called *trendline* which allows you to display various fits to data. A *linear* trendline is, in fact, a least squares fit. Unfortunately, this feature does not automatically display the parameters for the fit, so it's not as much use as it could be.

7.4 Procedure

7.4.1 Preparation

You are welcome to go ahead and do as much of the exercise on your own as you wish; you can just bring your spreadsheet with you to the lab and demonstrate the points indicated. If you get it all done in advance, that's great.

If you do it on your own, then print a copy of the spreadsheet showing *formulas* along with the ones indicated in the post-lab questions. (You only need to show the formulas for rows mentioned in the instructions.) Print the graphs as well.

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Pre-lab Tasks

PT1: Read over Chapter 4, *"Graphs and Graphical Analysis"*, and copy the Equations 4.3 to 4.14 to the appropriate places below. (For instance, copy Equation 4.4 to complete Equation 7.4.) *If you don't want to print the pages of the manual, copy Equations 4.3 to 4.14 to a piece of paper and bring it with you.*

PT2: Open the spreadsheet for this exercise from the web page. On the tabbed pages for "raw data" and the three linearizations, insert rows as needed and type in your linearized data for each of the 3 linearizations. Save the spreadsheet on a USB memory stick or in your WLU account to bring to the lab.

PT3: Rewrite the Equations 4.7, 4.8, 4.9, 4.10 for m_{max}, b_{min}, m_{min}, b_{max} for a line with *negative* slope.

$m_{min} =$

$b_{max} =$

PT4: Fill in the co-ordinates from **PT3** in Figure 7.2.

Figure 7.3: Spreadsheet page for raw data

Figure 7.4: Spreadsheet page for first linearization

Figure 7.5: Spreadsheet page for second linearization

Figure 7.6: Spreadsheet page for third linearization

7.4.2   Investigation

In-lab Tasks

In this exercise the in-lab tasks appear throughout the section.

Part 1: Plotting a Graph

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> **These instructions are for using Excel. If you are using LibreOffice, skip ahead to page 90.**

Setting up the spreadsheet (Excel instructions)   *Use the data which is already in the spreadsheet. Only change to your own linearized data after the formulas have all been set up correctly.*

1 Load the graph from the lab page.

2 Insert a chart in the box on the first tabbed page.
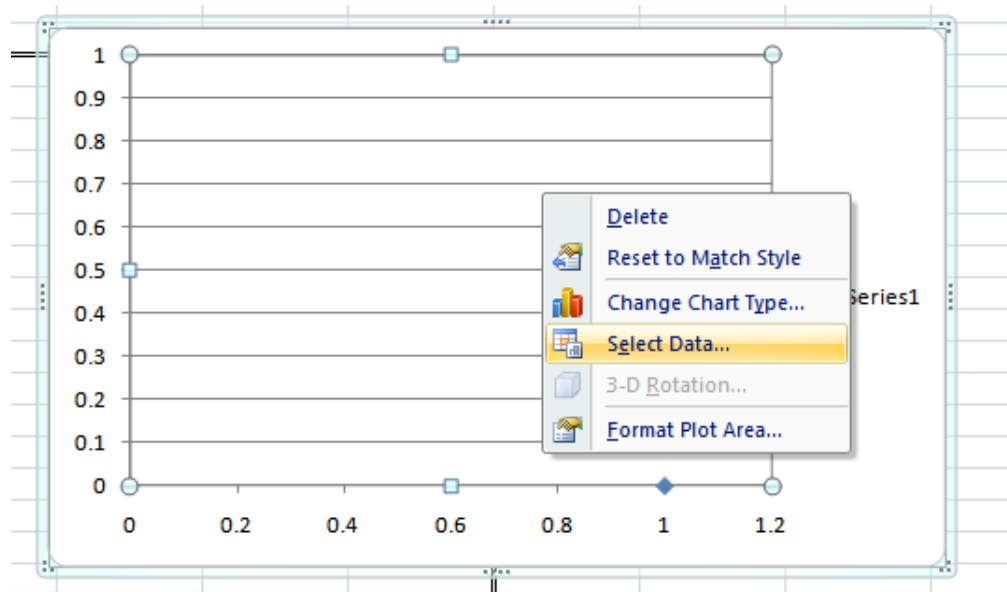
   • Make sure it is an xy (scatter) graph.

Figure 7.7: Adding a series to the graph: step 1

These instructions are for using Excel. If you are using LibreOffice, skip ahead to page 90.
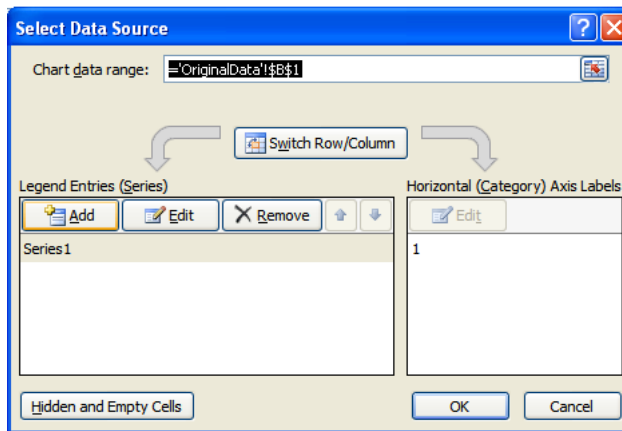

Figure 7.8: Adding a series to the graph: step 2

- The x series should be **C6** to **C9**.
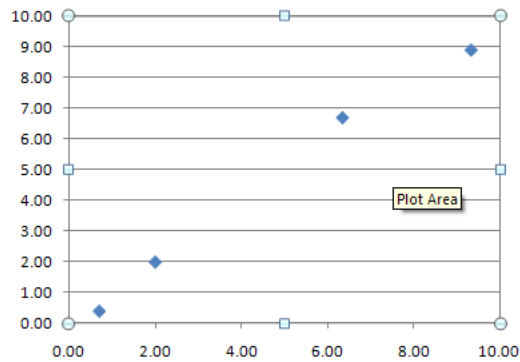- The y series should be **E6** to **E9**.

Figure 7.9: Basic graph without error bars

These instructions are for using Excel. If you are using LibreOffice, skip ahead to page 90.

3 Click on the *"Layout"* tab. You should now see the options for error bars.
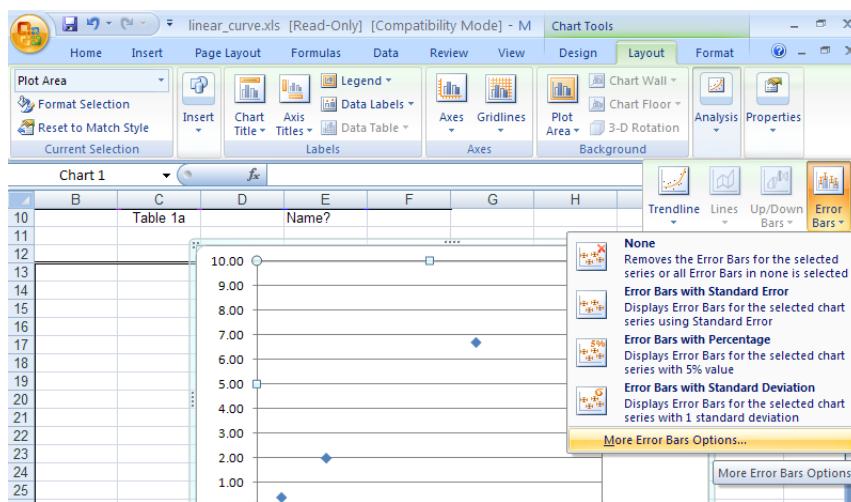
Figure 7.10: Adding vertical error bars

These instructions are for using Excel. If you are using LibreOffice, skip ahead to page 90.

- Note options for x and y error bars.
- Select y error bars, and pick *Custom*.
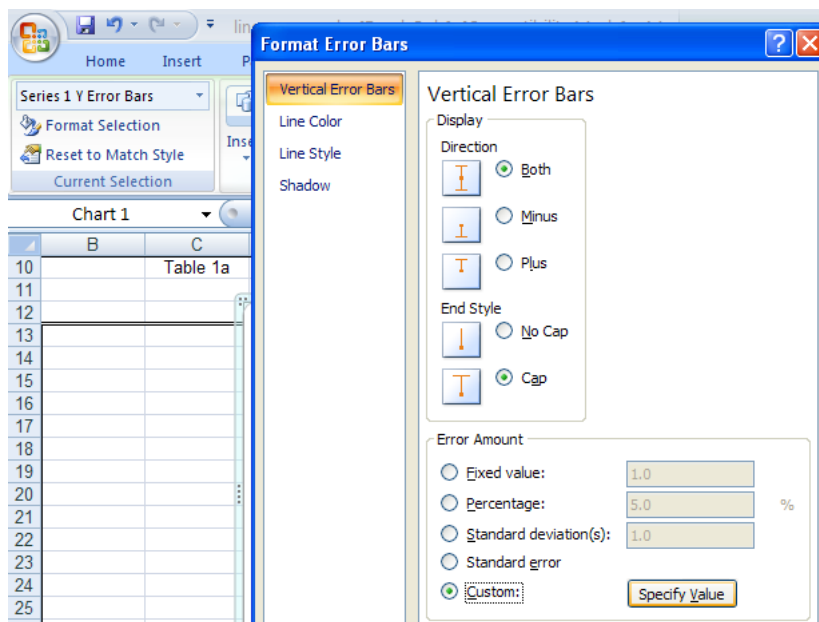
Figure 7.11: Custom vertical error bars

- Select series **F6** to **F9** for both **+** and **-**.

These instructions are for using Excel. If you are using LibreOffice, skip ahead to page 90.
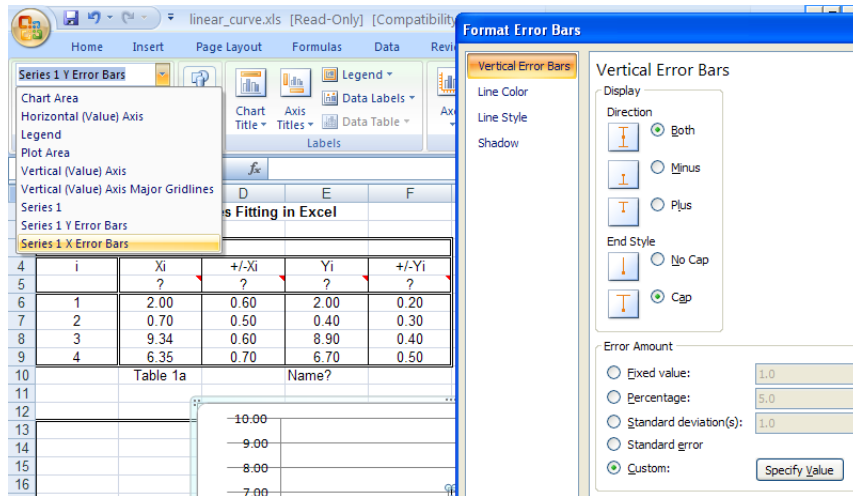
Figure 7.12: Adding horizontal error bars

- Repeat for x error bars, using **D6** to **D9**.

These instructions are for using Excel. If you are using LibreOffice, skip ahead to page 90.
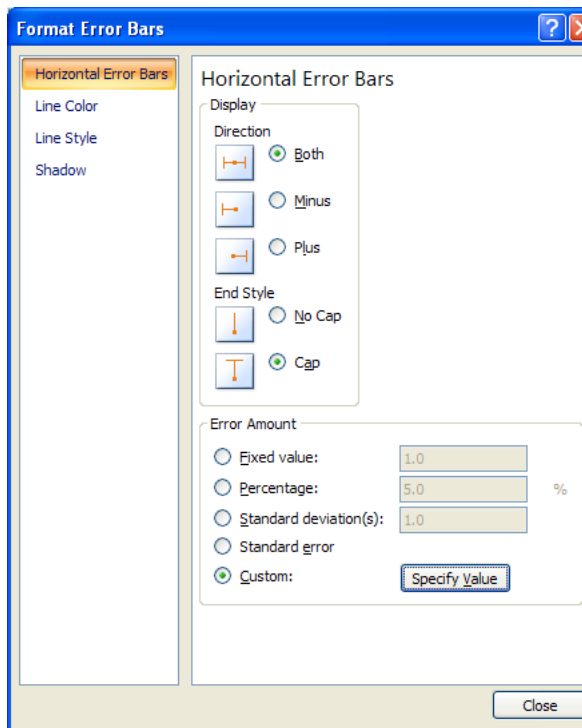
Figure 7.13: Custom horizontal error bars

End of Excel specific instructions for Part 1. Go to page 95.

These instructions are for using LibreOffice. If you are using Excel, go back to page 84.

Setting up the spreadsheet (LibreOffice instructions) *Use the data which is already in the spreadsheet. Only change to your own linearized data after the formulas have all been set up correctly.*

1 Load the graph from the lab page.

2 Insert a chart in the box on the first tabbed page.

  • Make sure it is an xy (scatter) graph.
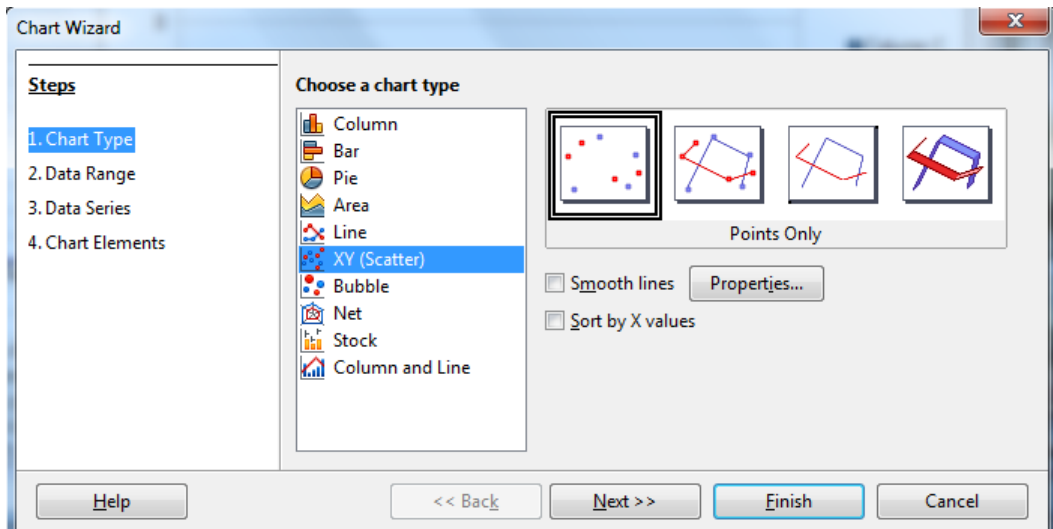
Figure 7.14: Choosing an xy graph

These instructions are for using LibreOffice. If you are using Excel, go back to page 84.

- Right click on the graph and select **Data Range** to select data.
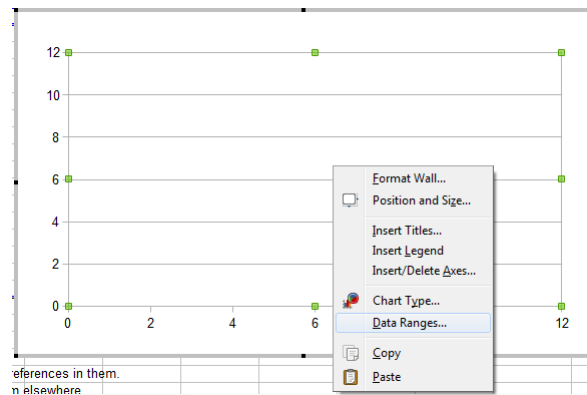
Figure 7.15: Adding a series to the graph: step 1

- Choose options as shown:
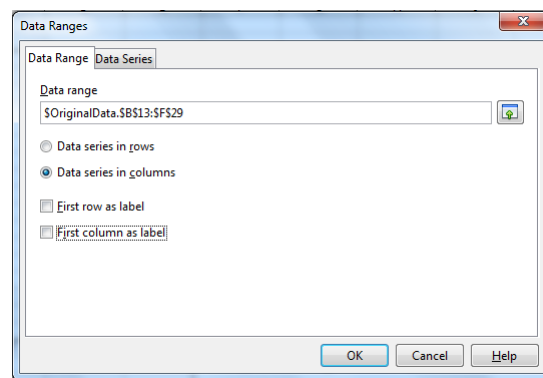
Figure 7.16: Series in columns, without labels

These instructions are for using LibreOffice. If you are using Excel, go back to page 84.
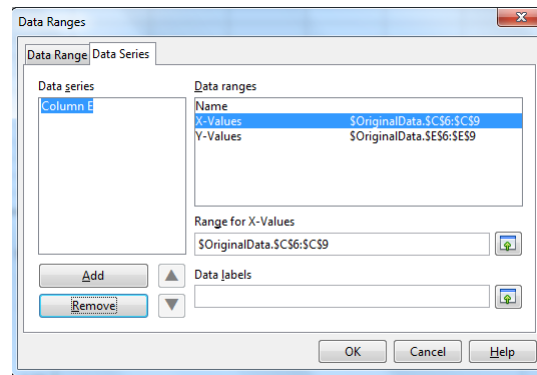
Figure 7.17: Adding a series to the graph: step 2

- The x series should be **C6** to **C9**.
- The y series should be **E6** to **E9**.
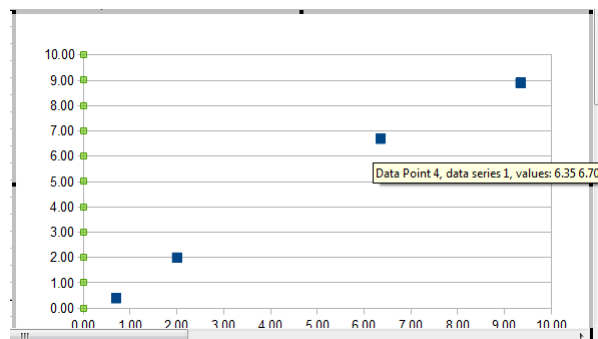
Figure 7.18: Basic graph without error bars

These instructions are for using LibreOffice. If you are using Excel, go back to page 84.

3 Right click on a data point to allow you to format the series.
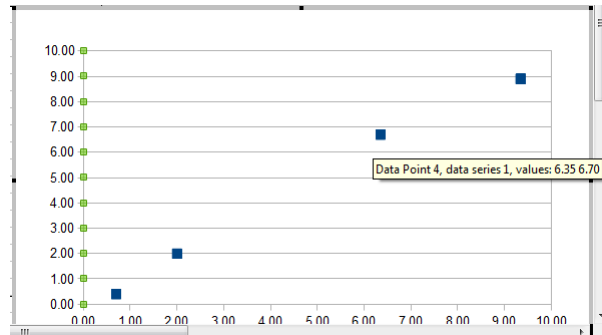
Figure 7.19: Adding vertical error bars

- Note options for x and y error bars.
- Select y error bars, and pick *Cell Range*.
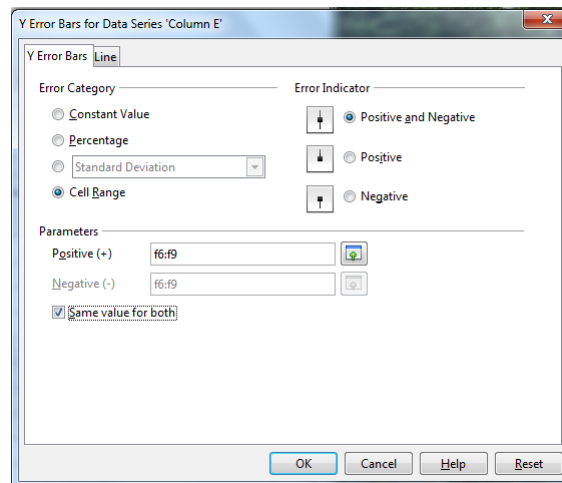
Figure 7.20: Using a cell range for vertical error bars

- Select series **F6** to **F9** and check **Same Value for both** .

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These instructions are for using LibreOffice. If you are using Excel, go back to page 84.
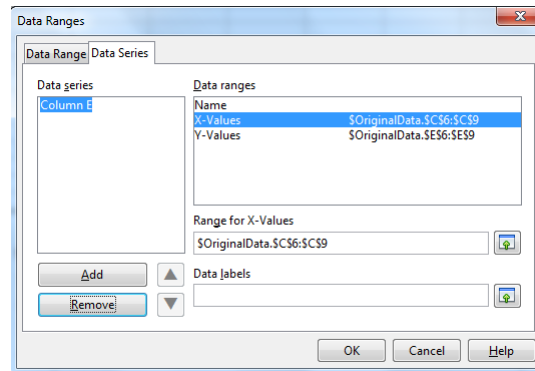
Figure 7.21: Adding horizontal error bars

- Repeat for x error bars, using **D6** to **D9**.
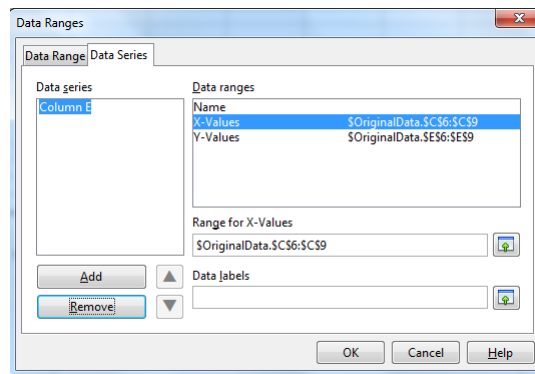
Figure 7.22: Using a cell range for horizontal error bars

End of LibreOffice specific instructions for Part 1

Common instructions for Part 1.

At this point there should *not* be a line connecting the data points. If there is, turn it off.

IT1: Demonstrate graph as is.
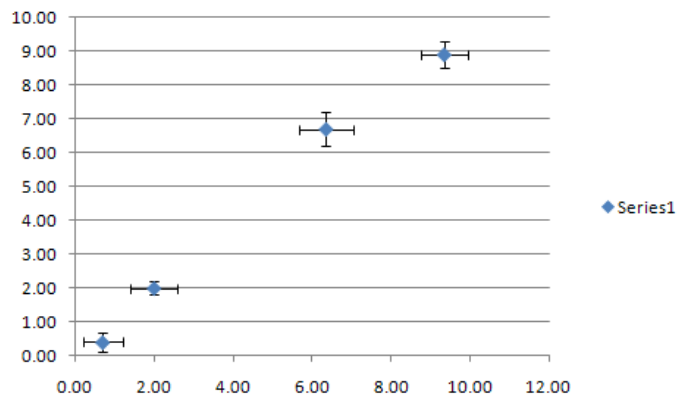
Figure 7.23: Graph with error bars (Excel)
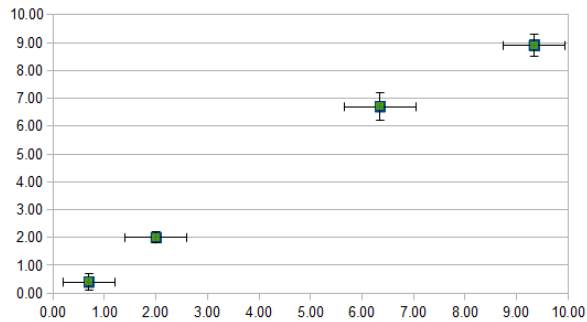
Figure 7.24: Graph with error bars (LibreOffice)

Part 2: Performing a Least Squares Fit Using Formulas

In this part, you're going to use formulas to do a least squares fit.

Remember that the least squares fit gives values for b, the y-intercept, and m, the slope, as follows:

$$b = \tag{7.3}$$

and

$$m = \tag{7.4}$$

For the linear case, S can be shown to have a value of

$$S = \tag{7.5}$$

$$\sigma = \tag{7.6}$$

where

$$\nu =$$
(7.7)

is the number of **degrees of freedom** mentioned earlier. (Often the symbol ν is used for degrees of freedom.)
Also, the **standard error** in the intercept is

$$\sigma_b =$$
(7.8)

and the standard error in the slope is

$$\sigma_m =$$
(7.9)

R^2 is defined as follows:

$$R^2 = \frac{\left(N \sum x_i y_i - (\sum x_i)(\sum y_i)\right)^2}{\left(N \sum x_i^2 - (\sum x_i)^2\right)\left(N \sum y_i^2 - (\sum y_i)^2\right)}$$

These instructions are for using Excel. If you are using LibreOffice, skip ahead to page 100.

(Excel instructions)

1 Go to the second tabbed page, and replace number cells for everything other than data values with formulas. (Make sure to use a function for N.)

	A	B	C	D	E	F	G	H	I
16								190.0003	
17									
18		Least Squares calculations (by formulae)							
19		Quantity				Value			
20		Slope				0.9936			
21		Y-intercept				-0.0681			
22		Sum of Squares Error (S)				0.36613			
23		Standard Deviation				0.42786			
24		Standard Error in Slope				0.06208			
25		Standard Error in Y-intercept				0.35669			
26		R^2				0.99225			
27				Table 2a	Name?				
28									

Figure 7.25: Least Squares Fit Using Formulas

2 Use the values of x_{min} and x_{max} in **C31** and **C32** and the equation of the line of best fit to calculate corresponding y values in **F31** and **F32** Add this series to your graph to show the line of best fit. Right click on one of the points, and format the data series. You want to get a line with no endpoints.

These instructions are for using Excel. If you are using LibreOffice, skip ahead to page 100.
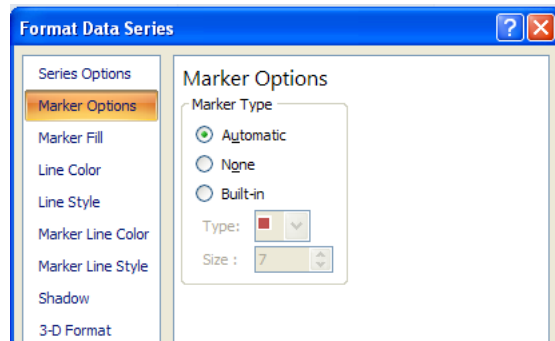
Figure 7.26: Marker options
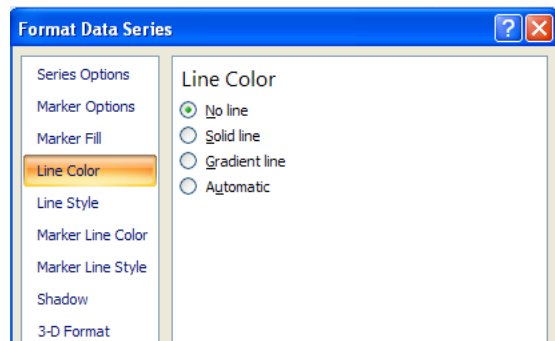
Figure 7.27: Line options

End of Excel specific instructions for Part 2 Go to page 102.

These instructions are for using LibreOffice. If you are using Excel, go back to page 98.

(LibreOffice instructions)

1 Go to the second tabbed page, and replace number cells for everything other than data values with formulas. (Make sure to use a function for *N*.)

	A	B	C	D	E	F	G	H	I
16								190.0003	
17									
18		Least Squares calculations (by formulae)							
19		Quantity				Value			
20		Slope				0.9936			
21		Y-intercept				-0.0681			
22		Sum of Squares Error (S)				0.36613			
23		Standard Deviation				0.42786			
24		Standard Error in Slope				0.06208			
25		Standard Error in Y-intercept				0.35669			
26		R^2				0.99225			
27				Table 2a	Name?				
28									

Figure 7.28: Least Squares Fit Using Formulas

2 Use the values of x_{min} and x_{max} in **C31** and **C32** and the equation of the line of best fit to calculate corresponding y values in **F31** and **F32** Add this series to your graph to show the line of best fit. Right click on one of the points, and format the data series. You want to get a line with no endpoints.

These instructions are for using LibreOffice. If you are using Excel, go back to page 98.
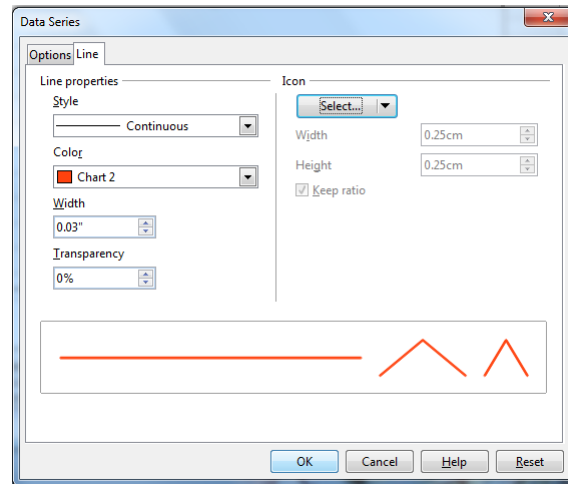
Figure 7.29: Line types

End of LibreOffice specific instructions for Part 2

Common instructions for Part 2.

IT2: Demonstrate graph as is.

Part 3: Performing a Least Squares Fit Using Built-in Features

> **These instructions are for using Excel. If you are using LibreOffice, skip ahead to page 105.**

(Excel instructions) Here you're going to identify the values produced by Excel's **LINEST()** function by comparing them to the values you got from the formulas.

1 Go to the third tabbed page, and use the **LINEST()** function to do a least squares fit.

- Put the function in **C18**, with *constant*=1 and *stats*=1.
- Highlight **C18** to **D22**.
- On a PC, press ⟨F2⟩ *followed by* ⟨CTRL⟩⟨SHIFT⟩⟨ENTER⟩.
  (On a MAC, click inside the formula editing box, and press
  ⟨COMMAND⟩⟨ENTER⟩.)
- *Fill in* **B18** *to* **B22** *and* **E18** *to* **E22** *with names from the previous page. (You don't have to fill in the ones that weren't previously calculated.)*[3]

---

[3]One of the quantities is the number of **degrees of freedom** mentioned earlier. It should be easy to identify. The two "extra" quantities produced are the Regression Sum of Squares given by

$$SSR = \left(\sum (y_i - \bar{y})^2 \right) - S$$

which can be shown to be given by

$$SSR = \left(\sum y_i{}^2 \right) - N\bar{y}^2 - S$$

and F given by

$$F = \frac{\frac{SSR}{\nu_R}}{\frac{S}{\nu}}$$

which you may find out about in a statistics class when discussing *Analysis of Variance*.

These instructions are for using Excel. If you are using LibreOffice, skip ahead to page 105.

	A	B	C	D	E	F
13						
14						
15			Output from LINEST			
16						
17						
18		?	0.99359843	-0.06806879	?	
19		?	0.06208041	0.35668978	?	
20		?	0.99225289	0.42786005	?	
21		?	256.160768	2	?	
22		?	46.8938716	0.36612844	?	
23						
24						
25			Table 2b	Name?		
26						

Figure 7.30: Least Squares Fit Using LINEST

End of Excel specific instructions for Part 3 Go to page 108.

These instructions are for using LibreOffice. If you are using Excel, go back to page 103.

(LibreOffice instructions) Here you're going to identify the values produced by LibreOffice's **LINEST()** function by comparing them to the values you got from the formulas.

1 Go to the third tabbed page, and use the **LINEST()** function to do a least squares fit.

- Put the function in **C18**, with *constant*=1 and *stats*=1.

- Use the function wizard.

Figure 7.31: LINEST using the function wizard

> **These instructions are for using LibreOffice. If you are using Excel, go back to page 103.**

- Choose the "Array" option to show all of the output values.
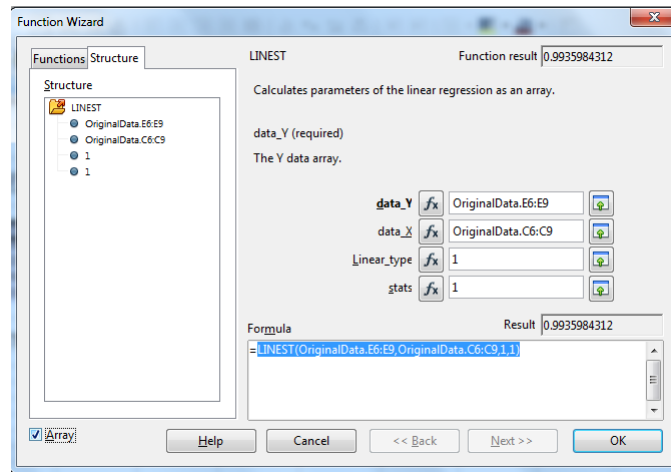
Figure 7.32: Choosing the "Array" option

- Click OK.
- Fill in **B18** to **B22** and **E18** to **E22** with names from the previous page. (You don't have to fill in the ones that weren't previously calculated.)[4]

---

[4]One of the quantities is the number of **degrees of freedom** mentioned earlier. It should be easy to identify. The two "extra" quantities produced are the Regression Sum of Squares given by

$$SSR = \left(\sum (y_i - \bar{y})^2 \right) - S$$

which can be shown to be given by

$$SSR = \left(\sum y_i^2 \right) - N\bar{y}^2 - S$$

and F given by

$$F = \frac{\frac{SSR}{\nu_R}}{\frac{S}{\nu}}$$

which you may find out about in a statistics class when discussing *Analysis of Variance*.

These instructions are for using LibreOffice. If you are using Excel, go back to page 103.

Figure 7.33: Least Squares Fit Using LINEST

End of LibreOffice specific instructions for Part 3

Common instructions for Part 3.

3 On the graph, select data series as before, and choose *linear trendline*. See that it fits on top of the best fit line from before, proving it is a least squares fit.

IT3: Demonstrate graph as is. Show that the linear trendline is actually a least squares fit.

Part 4: Finding Maximum and Minimum Slopes

*Note that on the "Small Scatter" tab of the spreadsheet, the values in cells **D20** to **D23** are **point numbers**, (i.e. point 1 is the first data point, etc.), and so the values in each of those cells must be integers, and they must be between 1 and N, since each one refers to a **data point number**.*

1 Go to fourth tabbed page, and put in reference to third page (i.e. page using **LINEST()**) to create meaningful tables.

2 Go to fifth tabbed page and add series for lines of maximum and minimum slope using **E20** to **F21** and **E22** to **F23**. Format the series like the line of best fit to have no endpoints.

3 Change points referenced in **D20** to **D23** to produce lines of maximum and minimum slope which cross all error bars.

4 Look at the cells calculating uncertainty in slope and y intercept and understand how they are calculated using the **index()** function.

5 Put in formulas in **C38**, **C39**, **E38**, **E39**, **F38**, **F39** to get endpoints of lines of maximum and minimum slope which go the full width of the graph. Replace these values for the series above. (**E20** to **F21** and **E22** to **F23**).

IT4: Demonstrate graph as is. Explain the determination of uncertainties in slope and *y*-intercept this way.

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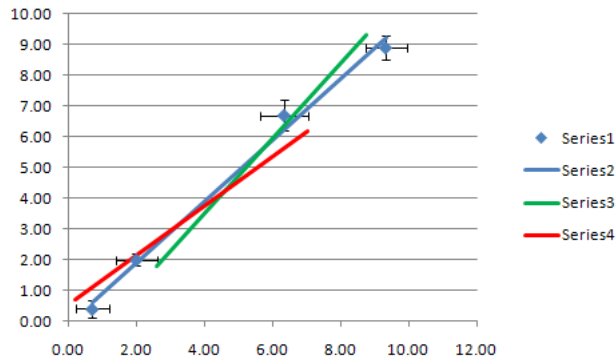
Figure 7.34: Adding lines of maximum and minimum slope
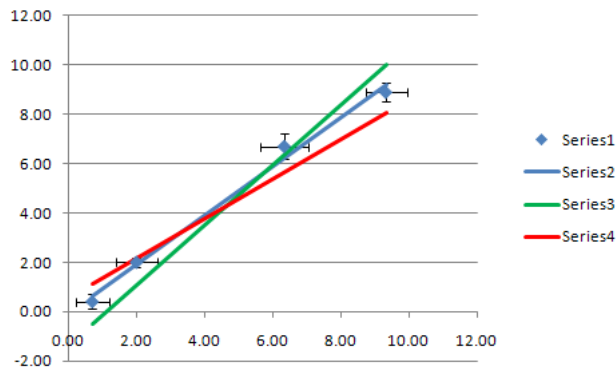
Figure 7.35: Full length lines of maximum and minimum slope

Once this spreadsheet is set up, you would only usually include the information from either the "small scatter" or the "large scatter" tab in a report; all of the other pages are just in order to make the exercise more organized. (Don't just print the spreadsheet page, but use it as a guide to figure out what information to include and how it should be organized.)

Part 5: Modifying the Spreadsheet for More Data Points

1 Insert rows as needed on each tab of the spreadsheet between rows 7 and 8 to allow as many data points as you have in your data from *"Standing Waves on a String"*.

2 Copy row 7 and paste it over the inserted rows and the row that was *previously* row 8.

3 Modify formulas on the inserted rows on any page where necessary to work correctly with this change. *If you've done things correctly, this shouldn't be necessary.*

IT5: Demonstrate spreadsheet as is. Show that the trial numbers and number of data points are correct.

7.4.3   Analysis

1. **If not done as a pre-lab task**, on the *"raw data"* tab, fill in your raw data from *"Standing Waves on a String"*.

2. The three tabs following the *"raw data"* tab are for each of the linearizations determined in the *"Exercise on Linearizing Equations"*. **If not done as a pre-lab task**, use the linearization formulas to fill in each of the tables with values calculated from the raw data.

3. For each linearization, replace the first page data with links to your linearization data.

4. For each linearization, adjust the "summary" tab as appropriate, based on whether the scatter is small or large.

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Post-lab Tasks

T1: Print off the "summary" tab for each of the three linearizations, making sure you make the appropriate changes for each in rows 4 and 5, and attach them to the lab report, along with the graphs of each of the linearizations.

7.5 Recap

By the end of this exercise, you should understand the following terms:
- linear graph

- error bars

- least squares fit

- correlation coefficient

- large scatter of data points

- small scatter of data points

In addition, you should, using a spreadsheet, be able to:
- plot a linear graph

- add error bars

- perform a least squares fit

- show the least squares fit line on the graph with the data
  You should also be able to

- determine whether the points on a graph classify as either "small" or "large" scatter, and calculate graphical uncertainties appropriately in either case;

- compare different linearizations of the same function and to explain why one may be preferred over others;

- suggest, based on your graphical uncertainties, whether or not having more precise measuring instruments would improve the results;

- suggest, based on the shape of your graph, whether there is evidence of systematic error in your data.

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7.6   Summary

Item	Number	Received	weight (%)
Pre-lab Questions	0	_____	0
In-lab Questions	0	_____	0
Post-lab Questions	0	_____	0
Pre-lab Tasks	4	_____	30
In-lab Tasks	5	_____	50
Post-lab Tasks	1	_____	20

Chapter 8

Exercise on Library Research

8.1 Purpose

The purpose of this exercise is to introduce you to the library and library research tools.

8.2 Introduction

Library research is an essential part of any research project.

8.3 Theory

In order for your results to be of any interest or relevance to anyone else, you need to establish things like

- *why* others have done similar research in the past,

    or

- *why* they haven't;

- *what* others have done in the past,

    and

- *what* kind of results they have produced.

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The library is the most important place to go for all of this information. Even though many journals are now available online, the library has the necessary tools and databases to actually allow you to *find* the kind of articles you're looking for.

> *If you find one article that's the kind of thing you're looking for, one way to look for more articles is to use the subject of that article for a further search. Another way is to look at the articles in the bibliography of that one.*

8.4   Procedure

8.4.1   Preparation

There are no pre-lab questions or tasks for this exercise.

8.4.2   Investigation

In-lab Questions

There are no specific questions to be completed during the lab time.

In-lab Tasks

IT1: Show that you have completed the in-lab assignment during the presentation.

   The lab time is free for you to use after the presentation is finished. Your task for the week is to follow the instructions given for each of the journal articles listed below.

Articles

Here are the specific articles you're going to investigate. You're given different information about each one to make the searches a bit different. The articles themselves are about topics which are pertinent to the labs or your university experience.

- A psychology article by Justin Kruger and David Dunning in 1999 about how people perceive their own abilities

- An article titled "Experimental Outcomes of Laboratory Instruction in Elementary College Physics" from the American Journal of Physics in 1952 studying what people learn from physics labs

- An article in "Physics in Canada" by Amy C. Rowat in 2005 about the attractiveness of physics to potential students

Using Online Databases

1. For each of the journal articles listed, use the online databases to find it.

2. Explain the searches you performed; specifically:

   - indicate which database you used,
   - list search criteria,
   - show list of results,
   - explain how you identified the correct article,
   - state whether the article is available online, or only in print.

Using Primo and/or databases

3. Search Primo and/or databases to try to find the journals.

4. Sate which are in the tri-university system (but not online). Explain the Primo or database search you used to find them.

5. Identify any journal which is not in the tri-university system (or online). Explain the Primo or database search you used to determine this.

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Research

6. For all articles identified:

- Print the bibliographic information for the article.(i.e. article name, author(s), journal name, volume and number, year published, page numbers).

- Print the first page of the article.

- In one or two sentences, state one of the major conclusions of the article in your own words. Highlight the location of that information on the first page of the article.

8.4.3 Analysis

Post-lab Questions

Q1 to **Q3:** Hand in the material required above for each article. Arrange the information by article, in the order of items above. (Each article counts as one question.)

8.5 Recap

By the end of this exercise, you should understand the following terms:

- refereed journal

- online database

In addition, you should be able to find

- journal articles

- books

using the library resources.

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8.6   Summary

Item	Number	Received	weight (%)
Pre-lab Questions	0	_____	0
In-lab Questions	0	_____	0
Post-lab Questions	3	_____	50
Pre-lab Tasks	0	_____	0
In-lab Tasks	1	_____	50
Post-lab Tasks	0	_____	0

Chapter 9

Exercise on Plotting Curves Using a Spreadsheet

9.1   Purpose

The purpose of this exercise is to become familiar with using a spreadsheet to produce curves of any function desired.

9.2   Introduction

Using a spreadsheet to graph data is convenient once you know what you're doing, and even plotting error bars and straight lines is relatively simple. However, once you need to plot data which is not linear, things become a bit more complicated.

9.3   Theory

Consider the data shown in Figure 9.1.

First of all, the graph does not have error bars. However it it also not smooth. *Dot-to-dot drawings do not usually belong in scientific reports; smooth curves are usually more appropriate.* The following discussion should help you to use a spreadsheet to produce non–linear plots.

Winter 2015
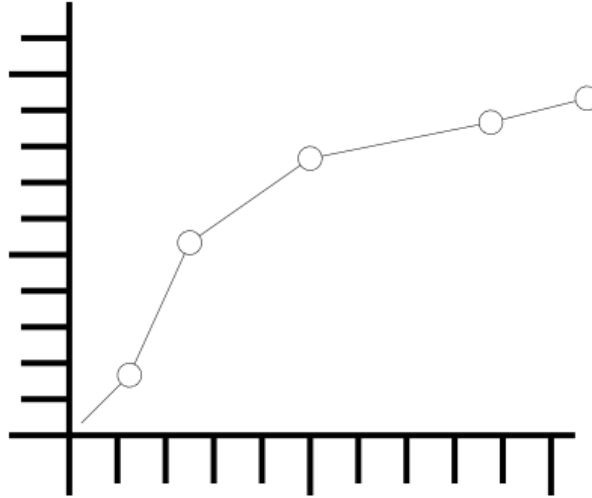
Figure 9.1: Logarithmic Fit the Wrong Way

9.3.1 Displaying Curves

You have previously seen how a a *line* can be drawn by simply connecting its endpoints. Similarly, a *curve* can be *approximated* by a series of very short line segments between points along the curve. If the points are close enough, the line will look smooth. If we wish to plot a curve in a spreadsheet, proceed as follows:

1. Instead of just using two values, x_{min} and x_{max} , for an x–series, create a series of values $x_i, i = 0 \dots N$ where

$$x_i = x_{min} + i \left(\frac{x_{max} - x_{min}}{N} \right)$$

so that

$$x_0 = x_{min}$$

and

$$x_N = x_{max}$$

and for each point

$$y_i = f(x_i)$$

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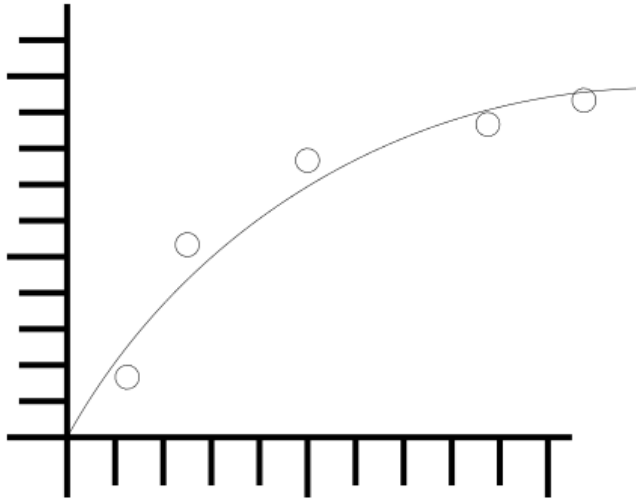
Figure 9.2: Logarithmic Fit the Right Way

Note that even though the x values for the first and last point are taken from the original data, the y values are not, since the purpose is to find points on the *curve*.

A value of 100 for N should be sufficient to make a smooth curve. If not, make it bigger. You can try making it smaller if that many points aren't needed.

2. For each of the x_i, calculate the corresponding y_i value from the curve equation and add this series to the graph.

3. Remove the markers and add lines for this series. All points in this series will thereby be joined with line segments. You should be able to produce a graph such as in Figure 9.2. (Error bars have been left off for simplicity. They can be produced in the usual way.)

Note that in the graph of Figure 9.2, none of the data points actually fall on the curve. This is often the case.

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9.3.2   Piecewise Defined Functions

In some cases, a curve must be made from data which must be fit to different equations in different regions. In this case the trick comes in trying to join the two fit equations smoothly.
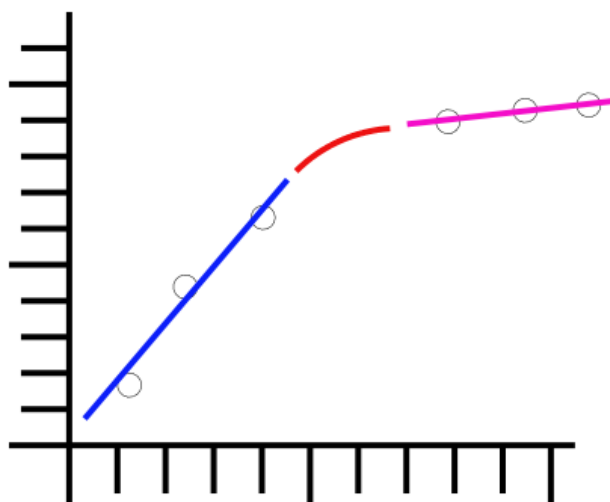
Figure 9.3: Piecewise Defined Function

In this case,

$$f(x) = \begin{cases} F_1(x), & \text{if } x \le x_1; \\ F_2(x), & \text{if } x \ge x_2. \end{cases}$$

where $x_2 > x_1$. To make the function $f(x)$ *continuous*, we require that

$$f(x_1) = F_1(x_1)$$

and

$$f(x_2) = F_2(x_2)$$

To produce a *smooth* fit, then in addition we require that

$$f'(x_1) = F_1{}'(x_1)$$

and

$$f'(x_2) = F_2'(x_2)$$

These 4 equations can be solved exactly by a polynomial in 4 unknowns, i.e. an equation of the form

$$f(x) = \alpha x^3 + \beta x^2 + \gamma x + \delta$$

and therefore

$$f'(x) = 3\alpha x^2 + 2\beta x + \gamma$$

To solve this, we need to set up a system of equations as follows: If we let

$$\mathcal{A} = \begin{pmatrix} x_1{}^3 & x_1{}^2 & x_1 & 1 \\ x_2{}^3 & x_2{}^2 & x_2 & 1 \\ 3x_1{}^2 & 2x_1 & 1 & 0 \\ 3x_2{}^2 & 2x_2 & 1 & 0 \end{pmatrix}$$

and

$$\mathcal{B} = \begin{pmatrix} F_1(x_1) \\ F_2(x_2) \\ F_1{}'(x_1) \\ F_2{}'(x_2) \end{pmatrix}$$

Then it should be clear that

$$\mathcal{A}X = \mathcal{B}$$

or

$$X = \mathcal{A}^{-1}\mathcal{B}$$

where

$$X = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}$$

($F_1'(x_1)$ can be determined by using x_1 and the fit point immediately to its left, and $F_2'(x_2)$ can be determined by using x_2 and the fit point immediately to its right.) This system can then be solved using the **matrix invert** and **multiply** features of a spreadsheet.

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Note: This will produce a *smooth* graph; whether it produces an *accurate* graph remains to be seen, (although smooth transitions are more common than others). This should produce a graph such as in Figure 9.3. Error bars have not been shown so that the smoothness of the curve can be seen easily. As before, error bars should be included unless you are specifically told to omit them.

9.4   Procedure

9.4.1   Preparation

Pre-lab Tasks

PT1: Bring the *completed* spreadsheet, (i.e. the one that you have filled in), from the the *"Exercise on Graphing and Least Squares Fitting using a Spreadsheet"* on a flash drive or save it in your WLU account.

PT2: Given T and ΔT, give the equation for $\Delta\sqrt{T}$. This was done in the *"Exercise on Linearizing Equations"*.

$\Delta\sqrt{T} =$

PT3: From the linearization, give the equations for μ and $\Delta\mu$ from the graph of λ vs. $\sqrt{T}$. This was done in the *"Exercise on Linearizing Equations"*.

$\mu =$

$\Delta\mu =$

9.4.2   Investigation

In-lab Tasks

Part 1: Plotting the points

The first step is to plot the points on a graph. The method for this should be familiar after the *"Exercise on Graphing and Least Squares Fitting using a Spreadsheet"*.

1. Load the spreadsheet from the lab page, and go to the appropriate tab.

2. Insert a graph in the box marked Graph 1 (non-linearized) on the page.

   - Make sure it is an xy (scatter) graph.
   - The x series should be **C5** to **C9**.
   - The y series should be **E5** to **E9**.

3. Add both x and y error bars.

   - Use series **D5** to **D9** for x error bars.
   - Use series **F5** to **F9** for y error bars.

IT1: Demonstrate the graph as is.

Part 2: Linearizing the equation

The next step is to linearize the data in order to find a fit equation. The method for this should be familiar after the *"Exercise on Graphing and Least Squares Fitting using a Spreadsheet"*.

1. Replace the numbers with formulas in **C14** to **F14** in Table 2 for linearized data. (The pre-lab task **PT2** should help you with the formulas for column **D**.) When they are correct, *copy* the formulas and *paste special* in **C15** to **C18** to finish the table.

2. In Table 3 use the *LINEST* function to do a least squares fit using the *linearized* data.

3. Insert a graph in the box marked Graph 2 (linearized)on the page.

   - Make sure it is an *xy* (scatter) graph.
   - The x series should be **C14** to **C18**.
   - The y series should be **E14** to **E18**.

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4. Add error bars to the graph.

   - Use series **D14** to **D18** for x error bars.
   - Use series **F14** to **F18** for y error bars.

5. Select the data series, and choose a linear trendline.

IT2: Demonstrate the graph as is.

Part 3: Creating points on the curve

1. Put references to the appropriate *LINEST* results in **D41**, **D42**, **F41**, and **F42** according to the linearization equations.

2. Note that formulas are already filled in for **D49** and **D50**, and a number of steps for the curve plot has been chosen. *Hint: In the linear fit, what are x and y?*

   From the linearization, we have the equation

   $$Y = mX + b$$

   So if we replace X and Y with the correct quantities, we get

   $$\_\_ = m\_\_ + b \tag{9.1}$$

   (Hint: You can get Y and X from **PT3** .)

3. Replace the value in **E56** with the formula using the results of the linearization. Don't forget the y-intercept.[1] Note that the value for λ is from the curve, not from the original data.

---

[1]Even when the y-intercept or the slope *should* be zero, use the actual value you get from the linearization for plotting the curve. The curve represents the fit you *got*, not necessarily the one you *expected*.

4. Put a formula in **C57** which can be copied to get a counter, if it hasn't been done already.

   Now, by generating several values for T and using Equation 9.1, you can generate a bunch of values which can be plotted on the original graph. This is the only new step.

5. In **D57**, put the formula for calculating the next value of T;

$$T_i = T_{min} + i\left(T_{max} - T_{min}\right)/N_{steps}$$

   (Be sure to use absolute references as appropriate so that this formula can be copied.)

6. *Copy* the formula in **E57** and *paste special* in **E58**.

7. *Copy* the formulas in **C57** to **E57**, and *paste special* in **C58** to **C70** to finish the table.

8. Add a series to your non-linear graph using the values from **D56** to **E71** to show the curve produced by the fit.

9. Format the data series to get a line with no endpoints.

IT3: Demonstrate the graph as is.

Part 4: Refining the curve

1. Try changing the number of points for the curve, by changing the number of steps and deleting or adding rows in Table 4 as needed. (When adding rows, you will need to *copy* formulas.)

IT4: Demonstrate graph with more points. Show how many are needed to make it smooth.

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9.4.3 Analysis

Post-lab Questions

There are no post-lab requirements since next week final marks will be calculated.

9.5 Recap

By the end of this exercise, you should understand:

- how a smooth curve can be plotted in a spreadsheet from any function

In addition, you should be able to:

- display any curve on a spreadsheet graph.

9.6 Summary

Item	Number	Received	weight (%)
Pre-lab Questions	0	_____	0
In-lab Questions	0	_____	0
Post-lab Questions	0	_____	0
Pre-lab Tasks	3	_____	30
In-lab Tasks	4	_____	70
Post-lab Tasks	0	_____	0

Appendix A

Information about Measuring Instruments

Fill this in as you use new measuring instruments so you will have a reliable reference. Put frequently used instruments in Table A.1 and experiment-specific ones in Table A.2.

ref. #	measuring instrument	precision measure	zero error	units
A1	stopwatch			
A2	GL100R balance			
A3	vernier caliper			
A4	micrometer caliper			
A5				
A6				
A7				

Table A.1: Measuring instrument information

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ref. #	measuring instrument	precision measure	zero error	units
B1				
B2				
B3				
B4				
B5				
B6				
B7				
B8				
B9				
B10				
B11				
B12				
B13				
B14				
B15				

Table A.2: Measuring instrument information (continued)

Appendix B

Derivation of the Least Squares Fit

Following is a simple derivation of the least squares fit.

Suppose the relationship between the two experimental parameters being studied is

$$y = f(x)$$

where x is the independent parameter which is varied, and y is the dependent parameter. If $f(x)$ is a polynomial function, or can be approximated by a polynomial, then the least squares method is a *linear* one, and it will almost always give reliable answers. If $f(x)$ cannot be expressed as a polynomial, but consists of transcendental functions, the least squares method is non-linear, and may or may not work reliably. In some cases, a change of variables may result in a polynomial, as in the exponential example above. A function like

$$y = a + \frac{b}{x} + \frac{c}{x^2}$$

is not a polynomial in x, but it is a polynomial in the variable $z = 1/x$.

Suppose the functional relationship between x and y is a polynomial of degree ℓ:

$$y = a_0 + a_1 x + a_2 x^2 \ldots a_\ell x^\ell \tag{B.1}$$

or

$$y = \sum_{j=0}^{\ell} a_j x^j \tag{B.2}$$

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and we have a set of N data points x_i, y_i obtained by experiment. The goal is to find the values of the $\ell + 1$ parameters $a_0, a_1 \ldots a_\ell$ which will give the best fit of Equation B.1 to our data points. The first piece of information to note is that

$$N \geq \ell + 1 \tag{B.3}$$

or else we will not be able to make a unique determination. For example, if $\ell = 1$, we need at least two data points to find the equation of the straight line. In order to make any meaningful statistical statements, however, we will need even more than $\ell + 1$ points, as we shall see later. A good rule of thumb: if we wish to fit our data with a polynomial of degree ℓ in a 95% confidence interval, we should choose N such that

$$N - (\ell + 1) \geq 10 \tag{B.4}$$

The idea behind the linear least squares method is to *minimize* the sum

$$S = \sum_{i=1}^{N} \left(y_i - \sum_{j=0}^{\ell} a_j x_i^j \right)^2 \tag{B.5}$$

S will be a minimum if

$$\frac{\partial S}{\partial a_k} = 0 \qquad k = 0, 1, 2 \ldots \ell \tag{B.6}$$

The result will be $\ell + 1$ linear equations in $\ell + 1$ unknowns:

$$\sum_{j=0}^{\ell} a_j \left(\sum_{i=1}^{N} x_i^{j+k} \right) = \sum_{i=1}^{N} x_i^k y_i \qquad k = 0, 1 \ldots \ell \tag{B.7}$$

which can be solved by standard matrix techniques for the unknown coefficients $a_0, a_1 \ldots a_\ell$. As an example, let us consider the case where $\ell = 1$, or

$$y = mx + b$$

In this case,

$$S = \sum_{i=1}^{N} (y_i - (mx_i + b))^2$$

Expanding Equation B.7, we have

$$b(N) + m \left(\sum_{i=1}^{N} x_i \right) = \sum_{i=1}^{N} y_i \tag{B.8}$$

$$b \left(\sum_{i=1}^{N} x_i \right) + m \left(\sum_{i=1}^{N} x_i^2 \right) = \sum_{i=1}^{N} x_i y_i \tag{B.9}$$

Then the intercept b and the slope m can be found from Cramer's rule

$$b = \frac{\left(\sum y_i \right) \left(\sum x_i^2 \right) - \left(\sum x_i \right) \left(\sum x_i y_i \right)}{N \left(\sum x_i^2 \right) - \left(\sum x_i \right)^2} \tag{B.10}$$

and

$$m = \frac{N \left(\sum x_i y_i \right) - \left(\sum x_i \right) \left(\sum y_i \right)}{N \left(\sum x_i^2 \right) - \left(\sum x_i \right)^2} \tag{B.11}$$

Appendix C

Lab Checklist

This marking checklist will be used for lab reports this term. You need to print one off and attach it to each lab report you hand in. Lab reports will be marked as follows:

- Start with 90

  For items *not* in italics

- Subtract 1 for each ˜.

- Subtract 2 for each − .

  For items *in* italics

- Subtract 3 for each ˜.

- Subtract 6 for each − .

Note the importance of items in italics. These are very important in a report, and so are weighted accordingly.

The other 10 marks will be based on how well the post-lab discussion questions were answered within the text of the report. *Remember that the answers to these questions should be an integral part of the report, not merely an afterthought.*

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Lab Format Checklist (V3.0g)

A. General

1.	Your own work	_____
2.	Complete	_____
3.	Clear and appropriate "Purpose"	_____
4.	Flows	_____
5.	Did not require help on or after due date	_____
6.	Correct grammar	_____
7.	Correct spelling	_____
8.	Complete sentences where required	_____
9.	Legible	_____
10.	Professionally presented	_____
11.	Properly identified (e.g. name, partner)	_____
12.	On time	_____
13.	Checklist included	_____
14.	Template included	_____

B. *Plagiarism Avoidance*

    1.   *Data only shared with partner(s)*   _____

    2.   *Individual choice of sample data*   _____

    3.   *Individual formatting*   _____

    4.   *Individual structure of text*   _____

C. Data

    1.   Neat   _____

    2.   Table column headings informative   _____

    3.   Units given   _____

    4.   Uncertainties given   _____

    5.   Reasonable values   _____

    6.   Reasonable uncertainties   _____

    7.   Correct number of significant figures   _____

    8.   Tables labeled (e.g. "Sample 1 Data")   _____

    9.   Tables given numbers (e.g. "Table #2")   _____

D. Calculations and Results

    1.   Any required derivations done correctly   _____

    2.   Analysis explained where needed   _____

    3.   Correct formulas used   _____

    4.   Sample calculations shown where needed   _____

    5.   All required values calculated   _____

    6.   Uncertainties included   _____

    7.   Units included   _____

    8.   Correct number of significant figures   _____

    9.   Appropriate use of standard form   _____

    10.   Theoretical or reasonable value   _____

    11.   Agreement of experiment with theory   _____

E. Linearizations
    1.   Independent (x) variable identified   _____
    2.   Dependent (y) variable identified   _____
    3.   Expected slope identified   _____
    4.   Expected y-intercept identified   _____

F. Graphs
    1.   Title meaningful   _____
    2.   Correct graph type and orientation   _____
    3.   Background colour and grid appropriate   _____
    4.   Plotting data in table   _____
    5.   Axis labels meaningful   _____
    6.   Correct axis units   _____
    7.   Points not connected   _____
    8.   Error bars in both dimensions or note if   _____
        too small
    9.   Error bars correct size   _____
    10.   Line of best fit shown without markers   _____
    11.   Number given (e.g. "Graph #3")   _____

G. Least Squares Fits
    1.   Points used for fit clearly identified   _____
    2.   Results given meaningful names   _____
    3.   Correct units for slope and intercept   _____
    4.   Correct indication of "large" or "small"   _____
        scatter
    5.   Reasonable points for maximum and min-   _____
        imum slopes if scatter "large"

H. *Error Discussion*
      1.   Sources listed are significant   _____
      2.   Sources are prioritized   _____
      3.   Systematic error consequences   _____
      4.   Evidence: i.e. test or bound   _____
      5.   Reasonable suggestions for improvement   _____

I. *Conclusions*
      1.   Relate to purpose   _____
      2.   Major results stated   _____
      3.   Comparisons made where appropriate   _____
      4.   Agreement noted when found   _____
      5.   % difference calculated only when no agree- _____
             ment

J. References
      1.   Source(s) of constants listed   _____

Appendix D

Marking Scheme

PC132 Lab and Exercise Weighting 2015 (Best adding to 100 will count)

A. Required

1.	MyLearningSpace Quizzes	10
2.	Linearizing Equations	10
3.	Graphing and Least Squares Fitting	20
4.	Library Research	10
5.	Curve Plotting	10
6.	Standing Waves on a String*[1]	10
7.	Standing Waves on a String report	30

B. Optional (Can replace other marks if higher)

1.	*Additional* weight of lab report, if over 50 %	10
2.	*Additional* weight of lab report, with writing centre consultation	10
3.	*Additional* weight of lab report, if over 80 %	10
4.	Lab test	20

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[1]except post-lab questions, which will be in the report

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Appendix E

Review of Uncertainty Calculations

E.1 Review of uncertainty rules

These are from the PC131 lab manual.

E.1.1 Repeated measurements

Arithmetic Mean (Average)

Note: In the following sections, each measurement x_i can be assumed to have an uncertainty pm, (i.e. the precision measure of the instrument used), due to measurement uncertainty.

The **arithmetic mean** (or average) represents the best value obtainable from a series of observations from "normally" distributed data.

$$\text{Arithmetic mean} \quad = \overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$
$$= \frac{x_1 + x_2 + \cdots + x_n}{n}$$

Standard Deviation

The **standard deviation** of a number of measurements is a measurement of the uncertainty in an experiment due to reproducibility. The standard deviation is given by

$$Standard\ Deviation = \sigma \quad = \quad \sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n - 1}}$$

$$= \quad \frac{1}{\sqrt{n-1}}\sqrt{\sum_{i=1}^{n} x_i^2 - \frac{(\sum_{i=1}^{n} x_i)^2}{n}}$$

With random variations in the measurements, about 2/3 of the measurements should fall within the region given by $\overline{x} \pm \sigma$, and about 95% of the measurements should fall within the region given by $\overline{x} \pm 2\sigma$. (If this is not the case, then either uncertainties were not random or not enough measurements were taken to make this statistically valid.)

This occurs because the value calculated for $\overline{x}$, called the **sample mean**, may not be very close to the "actual" **population mean**, μ, which one would get by taking an infinite number of measurements.

> Rule of thumb: For normally distributed data, an order of magnitude approximation for the standard deviation is 1/4 the range of the data. (In other words, take the difference between the maximum and minimum values and divide by 4 to get an approximate value for the standard deviation.)

Standard Deviation of the Mean

(In some texts this quantity is called the "standard *error* of the mean".) It is an *interval* around the *calculated* mean, $\overline{x}$, in which the *population* mean, μ, can be reasonably assumed to be found. This region is given by the *standard deviation of the mean*,

$$Standard\ deviation\ of\ the\ mean = \alpha = \frac{\sigma}{\sqrt{n}}$$

and one can give the value of the measured quantity as $\overline{x} \pm \alpha$. (In other words, μ should fall within the range of $\overline{x} \pm \alpha$.)

Uncertainty in the average

The uncertainty in the average is the *greater* of the uncertainty of the individual measurements, (i.e. *pm*, the precision measure of the instrument used), and α; i.e.

$$\Delta\overline{x} = \max\left(pm, \alpha\right)$$

If possible, when doing an experiment, enough measurements of a quantity should be taken so that the uncertainty in the measurement due to instrumental precision is greater than or equal to α. This is so that the random variations in data values at some point become less significant than the instrument precision. (In practice this may require a number of data values to be taken which is simply not reasonable, but sometimes this condition will not be too difficult to achieve.)

In any case, the uncertainty used in subsequent calculations should be the *greater* of the uncertainty of the individual measurements and α.

E.1.2   Rules for combining uncertainties

Basic arithmetic rules

The uncertainty in results can *usually* be calculated as in the following examples (if the percentage uncertainties in the data are small):

$$(a) \quad \Delta(A + B) = (\Delta A + \Delta B)$$
$$(b) \quad \Delta(A - B) = (\Delta A + \Delta B)$$
$$(c) \quad \Delta(A \times B) \approx |AB| \left(\left| \frac{\Delta A}{A} \right| + \left| \frac{\Delta B}{B} \right| \right)$$
$$(d) \quad \Delta(\frac{A}{B}) \approx \left| \frac{A}{B} \right| \left(\left| \frac{\Delta A}{A} \right| + \left| \frac{\Delta B}{B} \right| \right)$$

Note that the first two rules above *always* hold true.

Uncertainties in functions, by algebra

$$\Delta f(x) \approx |f'(x)| \, \Delta x \qquad\qquad\qquad (E.1)$$

Uncertainties in functions, by inspection

$$\Delta f(x) \approx f_{max} - f \qquad\qquad\qquad (E.2)$$

or

$$\Delta f(x) \approx f - f_{min} \qquad\qquad\qquad (E.3)$$

Sensitivity of Total Uncertainty to Individual Uncertainties

If $f = f(x, y)$, then to find the proportion of Δf due to each of the individual uncertainties, Δx and Δy, proceed as follows:

- To find Δf_x, let $\Delta y = 0$ and calculate Δf.

- To find Δf_y, let $\Delta x = 0$ and calculate Δf.

Uncertainties and Final Results

Always express final results with *absolute* uncertainties.

Mathematically, if two quantities a and b, with uncertainties Δa and Δb are compared, they can be considered to agree within their uncertainties if

$$|a - b| \leq \Delta a + \Delta b \qquad \text{(E.4)}$$

A value with no uncertainty given can be assumed to have an uncertainty of zero.

If two numbers do not agree within experimental error, then the *percentage difference* between the experimental and theoretical values must be calculated as follows:

$$Percent\ Difference = \left| \frac{theoretical - experimental}{theoretical} \right| \times 100\%$$
$$\text{(E.5)}$$

Remember: Only calculate the percent difference if your results do <u>not</u> agree within experimental error.

Significant Figures in Final Results

Always quote final answers with one significant digit of uncertainty, and round the answers so that the least significant digit quoted is the uncertain one.

E.2   Discussion of Uncertainties

- Spend most time discussing the factors which contribute most to uncertainties in your results.

- Always give a measured value or a numerical bound on an uncertainty.

- State whether any particular factor leads to a systematic uncertainty or a random one. If it's systematic, indicate whether it would tend to increase or decrease your result.

E.2.1   Types of Errors

- Measurable uncertainties

- Bounded uncertainties

- Blatant filler

Don't use *"human error"*; it's far too vague.

E.2.2   Reducing Errors

A. Avoid mistakes.

B. Repeat for consistency, if possible.

C. Change technique

D. Observe other factors as well; including ones which you may have assumed were not changing or shouldn't matter.

E. Repeat and do statistical analysis.

F. Change equipment; the last resort.

E.2.3   Ridiculous Errors

Anything which amounts to a mistake is not a valid source of error. A serious scientist will attempt to ensure no mistakes were made before considering reporting on results.

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Appendix F

Determining Graph Scatter

F.1   Introduction

You may have wondered if it's possible to determine automatically whether a linear graph has a "small" or a "large" scatter. Remember that the scatter is considered "small" if a line can be drawn that crosses the error bars for each data point. If no such line can be drawn, the scatter is considered "large". Consider a data point near the line of best fit. A point is considered "close"

$$y = mx + b$$

$$(x_i, y_i)$$
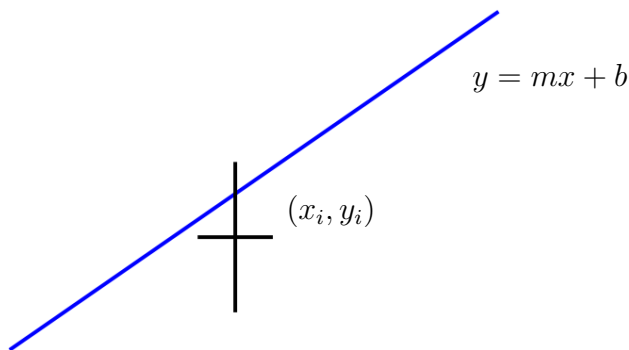
Figure F.1: Data point near best fit line

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to the line if it passes within the box formed by the error bars for the point.
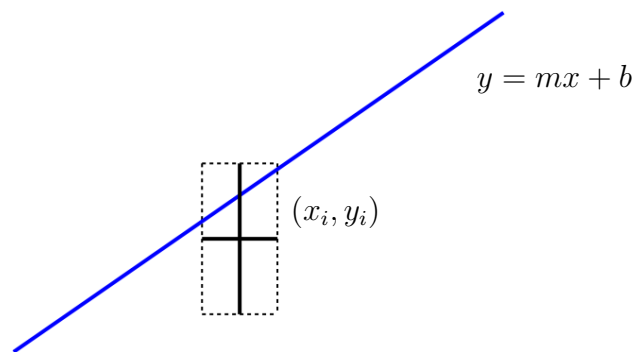
Figure F.2: Data point near best fit line with error "box"

Note that it's possible that a line could cut through one corner of the box without actually crossing one of the error bars.
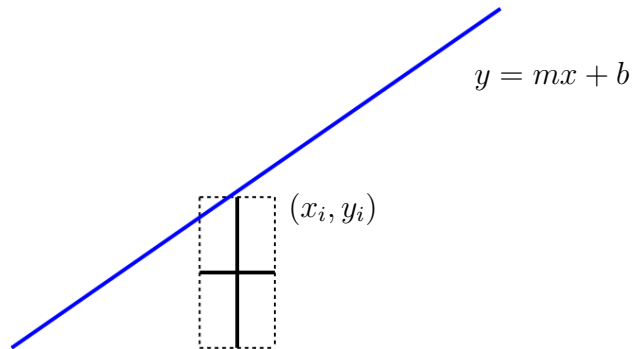
Figure F.3: Line inside "box" but missing error bars

Let's take a closer look at a data point. For point i, the error bars form a box with two diagonals.
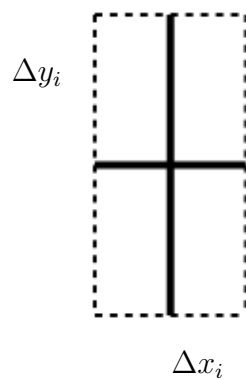
Figure F.4: Data point close-up view

We can redraw the "error box" showing the diagonals.
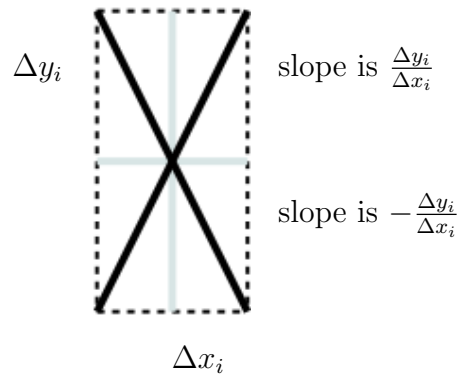
Figure F.5: Diagonals from error bars

Now it should be clear that if a line passes through this box, *even if it doesn't cross one of the error bars*, then it must cross one of these diagonals with an x value between $x_i + \Delta x_i$ and $x_i - \Delta x_i$.
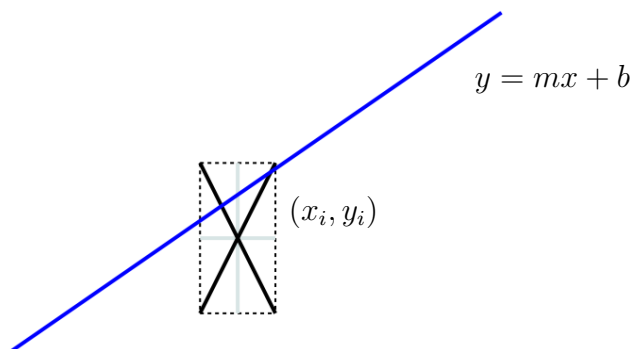
Figure F.6: Test whether line passes within error bars

The slope of the diagonal with the positive slope is

$$m_{+i} = \frac{\Delta y_i}{\Delta x_i} \tag{F.1}$$

The y-intercept of the diagonal with the positive slope is

$$b_{+i} = y_i - m_{+i}x_i \tag{F.2}$$

since, by definition, (x_i, y_i) must be on the line. Similarly, The slope of the diagonal with the negative slope is

$$m_{-i} = -\frac{\Delta y_i}{\Delta x_i} \tag{F.3}$$

The y-intercept of the diagonal with the negative slope is

$$b_{-i} = y_i - m_{-i}x_i \tag{F.4}$$

since, by definition, (x_i, y_i) must be on the line. For two lines $y = m_1 x + b_1$ and $y = m_2 x + b_2$, they cross when there is a single point (x_c, y_c) that satisfies both equations. Thus

$$y_c = m_1 x_c + b_1$$

and

$$y_c = m_2 x_c + b_2$$

Combining this gives

$$m_1 x_c + b_1 = m_2 x_c + b_2$$

and so

$$x_c = \frac{b_2 - b_1}{m_1 - m_2} \tag{F.5}$$

To see where the best fit line $y = mx + b$ crosses one of the diagonal lines, we just use Equation F.5 and so

$$x_{ci+} = \frac{b - b_{+i}}{m_{+i} - m} \tag{F.6}$$

and

$$x_{ci-} = \frac{b - b_{-i}}{m_{-i} - m} \tag{F.7}$$

If either of x_{ci+} or x_{ci-} is between $x_i + \Delta x_i$ and $x_i - \Delta x_i$ then the best fit line crosses the error bars for point i. If this is true for *all* of the data points, then this is a case of small scatter.

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F.1.1   When this doesn't work

This tests whether a *given* line crosses all of the error bars. So, if the least squares fit line crosses all of the error bars, then this test is easy to use. It's possible, though, that the least squares fit line *doesn't* cross all of the error bars, but another line can be drawn which does. Unless we can determine the equation of that line, we can't apply this test.

Another case where this doesn't work

If the x co-ordinate has no uncertainty, (i.e. there are only error bars in the y direction), then this won't work either. However, in that case, it easy to apply a similar test to see if the line crosses all of the (vertical) error bars. In this case, for each x_i, it must be that

$$y_i - \Delta y_i \leq mx_i + b \leq y_i + \Delta y_i$$

or

$$|y_i - (mx_i + b)| \leq \Delta y_i$$

Bibliography

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