Chapter 7

Exercise: Linearizing Equations

7.1 Purpose

The purpose of the exercise is to develop skills in producing linear graphs from various types of data and extracting results.

7.2 Introduction

This exercise will develop skills in linearizing data, so that a variety of relationships can be graphed as straight lines.

7.3 Theory

Often, the point of a scientific experiment is to try and find empirical values for one or more physical quantities, given measurements of some other quantities and some mathematical relationship between them. For instance, given a marble has a mass of 5gm, and a radius of 0.7cm, the density of the marble can be calculated given that \( v = \frac{4}{3}\pi r^3 \) and \( \rho = \frac{m}{v} \). (For the sake of simplicity, uncertainties will be ignored for now, although the calculation of those should be familiar by now.)

Many times, however, rather than having one measurement of a quantity, or set of quantities, we may have several measurements which should all follow the same relationships, (such as if we had several marbles made of the same material in the example above), and we wish to combine the
results. The usual way of combining results is to create a graph, and extract information (such as the density) from the slope and y–intercept of the graph.

One may be tempted to ask why a graph should be better than merely averaging all of the data points. The answer is that an average is completely unbiased. The variation of any one point from the norm is no more or less important than the variation of any other point. A graph, however, will show any point which differs significantly from the general trend. Analysis of the graphical data (such as with a least squares fit) will allow such “outliers” to be given either more or less weight than the rest of the data as the researcher deems appropriate. Depending on the situation, the researcher may wish to verify any odd point(s), or perhaps the trend will indicate that a linear model is insufficient. In any case, it is this added interpretive value that a graph has which makes it preferable.

If the data fits an equation of the form \( y = mx + b \), then it is easy to plot a straight line graph and interpret the slope and y–intercept, but it is rarely that simple. In most cases, the equation must be modified or linearized so that the variables plotted are different than the variables measured but produce a straight line.

**Linearizing equations** is this process of modifying an equation to produce new variables which can be plotted to produce a straight line graph. In many of your labs, this has been done already.

Look again at \( y = mx + b \). Note that \( y \) and \( x \) are variables, (as each can take on a range of values), while \( m \) and \( b \) are constants, (as there is only one value for each for all of the data points). We can linearize an equation if we can get it in the form

\[
\text{variable}_1 = \text{constant}_1 \times \text{variable}_2 + \text{constant}_2
\]

There are a few things to note:

1. Several constants combined together produces another single constant.

2. Powers or functions of constants are also constants.

3. Constants may have “special” values of 0 or 1 so they appear “invisible”. For example

\[
y = mx
\]
is still the equation of a straight line, where \( b = 0 \). As well,

\[
y = b
\]

is the equation of a line where \( m = 0 \).

4. Variables may be combined together to form new variables.

5. Powers or functions of variables are also variables.

7.3.1 Techniques for Linearization

If a relationship involves only multiplication and division, (including powers), then logarithms can be used to linearize. Sometimes taking roots or powers of both sides of an equation will help.

7.3.2 Procedure for Linearization

The steps are as follows:

1. Rearrange the equation to get one variable (or a function of it) on the left side of the equation; this becomes your \( y \) variable.

2. Regroup the right side of the equation to create a term containing the other variable (or some function of it).

3. Use the left-side variable (or the function of it) as your \( x \) variable, and then your slope should be whatever multiplies it; your \( y \) intercept is whatever additive term is left over.

Note: It is important to realize that you don’t need to understand an equation to linearize it; all you have to know is which parameters are variables (ie. things you have data for), and which parameters are constants (ie. things you want to calculate). Of course different experiments involving the same relationship may make different parameters variable, and so how an equation is linearized will depend on the data used.

To again consider the above example: The original equations were

\[
v = \frac{4}{3} \pi r^3
\]  
(7.1)

and

\[
\rho = \frac{m}{v}
\]  
(7.2)
where the quantities $m$ and $r$ are measured. (ie. We have several marbles of the same material, so we can get several measurements of $m$ and $r$, but we expect $\rho$ to be the same for all of them.) Thus for this situation, $m$ and $r$ are variables, and $\rho$ is a constant.

We can combine the two equations to get

$$\rho = \frac{m}{(4/3)\pi r^3} \quad (7.3)$$

or

$$\rho = \frac{3m}{4\pi r^3} \quad (7.4)$$

This equation has a constant on one side, and a mixture of variables and constants on the other. First we should rearrange it to get a variable on the left hand side. Suppose we rearrange the equation, giving

$$m = (4/3)\pi \rho r^3 \quad (7.5)$$

This leaves a variable on the left. From this point on, there are two main possibilities for how to proceed:

**Method I**

Now we can create a new variable, $Y$ such that

$$Y = m$$

By the rule about powers of variables being variables, then we can create a new variable $X$ given by

$$X = r^3$$

Then equation 7.5 above becomes

$$Y = (4/3)\pi \rho X \quad (7.6)$$

since $\pi$ is a constant, and $\rho$ should be, and using the rule that combinations of constants produce constants, then we can define $M$, a constant, (not the same as $m$), as

$$M = (4/3)\pi \rho$$

so equation 7.6 becomes

$$Y = MX + 0$$
which is the equation of a straight line. (In the case, \( B \), the \( y \)-intercept is zero.)\(^1\) So if we plot our “modified” variables, we should get a straight line, passing through the origin with a slope \( M \). How can we get \( \rho \) from the graph? Well, from above
\[
M = \frac{4}{3}\pi \rho
\]
so
\[
\rho = \frac{3M}{4\pi}
\]
where \( M \) is the slope of the graph.

**Method II**

We can take logarithms of both sides, so that \( Y \) such that equation 7.5 above becomes
\[
\ln m = \ln \left( \left(\frac{4}{3}\pi\rho\right) + \ln r^3 \right)
\]
(7.7)
grouping the terms so one only contains constants (and so the combination should be constant) and one only contains the variable \( r \). We can bring down the exponent so equation 7.7 becomes
\[
\ln m = \ln \left( \left(\frac{4}{3}\pi\rho\right) + 3 \ln r \right)
\]
Now we can create new variables, \( Y \) such that
\[
Y = \ln m
\]
and
\[
X = r
\]
which is the equation of a straight line. So if we plot our “modified” variables, we should get a straight line. How can we get \( \rho \) from the graph? Well, from above
\[
B = \ln \left( \left(\frac{4}{3}\pi\rho\right) \right)
\]
so
\[
\rho = \frac{3}{4\pi} e^B
\]
where \( B \) is the \( y \)-intercept of the graph. (In this case, the value you get from the graph for the slope should suggest whether the fit is a good one.)

\(^1\)Occasionally we can get a situation where the slope is similarly “invisible”, if it is 1 or 0.
7.3.3 Choosing a Particular Linearization

Often there may be more than one linear form for the equation so there may be more than one “right answer”. In this case, usually the preferable one is that which most simplifies understanding the graph or interpreting the results. For instance, in the above example, it would have been possible to use \((4/3)\pi r^3\) instead of \(r^3\) as our \(x\) variable, but that would make confusing axis scales and/or units (although it would have made the slope be \(\rho\) with no calculation).

7.3.4 Uncertainties in Results

After determining how equation parameters relate to graphical quantities as above, uncertainties can be determined as usual. In the above example Method I gives

\[ \Delta \rho = \frac{3\Delta M}{4\pi} \]

while for Method II

\[ \Delta \rho = \frac{3}{4\pi} e^B \Delta B \]

or

\[ \Delta \rho = \rho \Delta B \]

7.4 Procedure

1. The period of oscillation of a simple pendulum is

\[ T = 2\pi \sqrt{\frac{L}{g}} \]

where \(L\) and \(T\) are measured variables. Determine \(g\).

There are 3 different ways of doing this. In each case, show what should be plotted, what the slope and \(y\)-intercept will be, and how \(g\) and \(\Delta g\) come from the slope and \(y\)-intercept and their uncertainties.

(a) Plot \(T\) vs. ??
(b) Plot \(T^2\) vs. ??
(c) Plot \(\ln(T)\) vs. ??

Which of the above choices would you prefer and why?
7.5 Bonus: Do some of these

In each of the following questions, state the modified variables to be plotted, and state how the unknown(s) may be determined from the graph.

1. The position of a body starting from rest and subject to uniform acceleration is described by
   \[ s = \frac{1}{2}at^2 \]
   \( s \) and \( t \) are measured variables. Determine \( a \).

2. The fundamental frequency of vibration of a string is given by
   \[ \nu = \frac{1}{2l} \sqrt{\frac{T}{m}} \]
   \( \nu \), \( l \), and \( T \) are measured variables. Determine \( m \). (Hint: in this case one of your “modified variables” may incorporate two of the measured variables.)

3. The fundamental wavelength of vibration of a string is given by
   \[ \lambda = \frac{1}{\nu} \sqrt{\frac{F}{\mu}} \]
   \( \lambda \) and \( F \), are measured variables and \( \mu \) is a constant. Determine \( \nu \).

4. The Doppler shift of frequency for a moving source is given by
   \[ f = f_0 \frac{v}{v - v_0} \]
   \( f \) and \( v_0 \) are measured variables, \( f_0 \) is fixed and known. Determine \( v \).

5. The impedance of a series RC circuit is
   \[ Z = \sqrt{R^2 + \frac{1}{\omega^2C^2}} \]
   \( Z \) and \( \omega \) are measured variables. Determine \( R \) and \( C \).

6. The conductivity of an intrinsic semiconductor is given by
   \[ \sigma = Ce^{-\frac{E_g}{kT}} \]
   \( \sigma \) and \( T \) are measured variables. Determine \( E_g \) and \( C \).
7. The relativistic variation of mass with velocity is

\[ m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\( m \) and \( v \) are measured variables. Determine \( m_0 \) and \( c \).

8. The refraction equation is

\[ \mu_1 \sin \theta_1 = \mu_2 \sin \theta_2 \]

\( \theta_1 \) and \( \theta_2 \) are measured variables; \( \mu_1 \) is fixed and known. Determine \( \mu_2 \).