Uncertainty Calculations - Multiplication
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For the following examples, the values of $x = 2 \pm 1$ and $y = 32.0 \pm 0.2$ will be used.

Multiplication by a constant

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Multiplication by a constant with uncertainties

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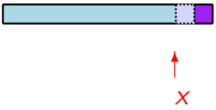
since $2 + 1 = 3$

so $4x = 8 \pm 4 = (4 \times 2) \pm (4 \times 1)$

Graphically,

The nominal value of x is here. (i.e. the value without considering uncertainties)

Graphically,



If we multiply by $1/2$, both x and Δx get smaller.

To summarize,

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When multiplying by a constant, we multiply the uncertainty by the constant as well.

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What if both numbers have uncertainties?

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since x can be as *big* as 3

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Just remember that uncertainties can be in either direction.

Graphically,

This is the nominal value of x

Graphically,

The maximum value of x includes Δx.

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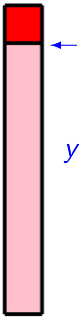
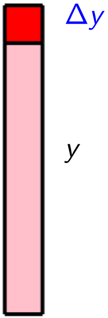


This is the nominal value of y.

Graphically,



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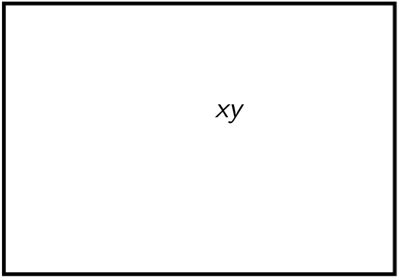
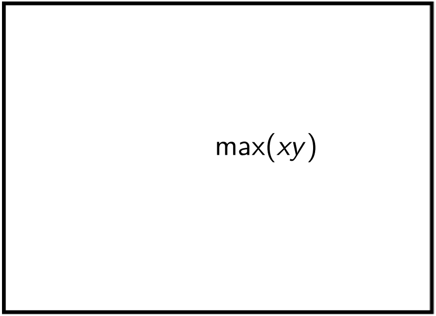


This is the nominal value of the area; i.e. xy.

Graphically,



This is the maximum value of the area; i.e. $(x + \Delta x) \times (y + \Delta y)$.

Graphically,

This is the difference between the nominal value of the area and the maximum value of the area.

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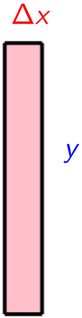

This part of the difference has a size of $y\Delta x$

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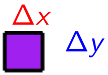

This part of the difference has a size of $\Delta x \Delta y$

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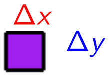

Because this is relatively small, we'll ignore it.

Graphically,

This is approximately the difference, and has a size of $y\Delta x + x\Delta y$

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Remember that if x or y can be negative, we'll need absolute value signs around the appropriate terms, since uncertainty contributions should always be given as positive numbers.

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2. When multiplying numbers, we add the *proportional* uncertainties.

$$\begin{aligned}(2 \pm 1) \times (32.0 \pm 0.2) &= (2 \times 32.0) \pm (2 \times 32.0)\left(\frac{1}{2} + \frac{0.2}{32.0}\right) \\ &= 64.0 \pm 64.0\,(0.5 + 0.00625) \\ &= 64.0 \pm 32.4\end{aligned}$$

Recap - continued

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3. Uncertainties in final results are usually expressed to one significant figure, so the above result becomes

$$64.0 \pm 32.4 = 60 \pm 30$$