Uncertainty Calculations - Division
Wilfrid Laurier University

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Calculations with uncertainties

When quantities with uncertainties are combined, the results have uncertainties as well. Following is a discussion of inversion and division.

For the following examples, the values of $x = 2 \pm 1$ and $y = 32 \pm 0.2$ will be used.
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Following is a discussion of **inversion** and **division**.
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Following is a discussion of inversion and division.

For the following examples, the values of $x = 2 \pm 1$ and $y = 32.0 \pm 0.2$ will be used.
Inversion with uncertainties
Inversion - Example

If we take the inverse of one of these numbers, $z = \frac{1}{y} = \frac{1}{32.0 \pm 0.2}$. $z$ can be as small as $1/32.2 = 1/32.0 + 0.2 \approx 0.03106$ since $y$ can be as big as $32.2$. $z$ can be as big as $1/31.8 = 1/32.0 - 0.2 \approx 0.03144$ since $y$ can be as small as $31.8$. 

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To summarize,

\[ z = \frac{1}{32} = 0.03125 \]

The nominal value of \( z \) is:

\[ z = 0.03125 \]

So we can say:

\[ z \approx 0.03125 \pm 0.00019 \]

and we see that:

\[ \Delta z \approx 0.00019 = \left( \frac{0.2}{y} \right) \left( \frac{\Delta y}{y} \right) \]

The proportional uncertainty in the inverse of a number is the same as the proportional uncertainty in the number.
To summarize,

\[ z \text{ can be as small as } \frac{1}{32.2} = \frac{1}{32.0+0.2} \approx 0.03106 \]
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So in general, \( \Delta \frac{1}{y} = \frac{1}{y} \left( \frac{\Delta y}{y} \right) \)
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So in general, \( \Delta \frac{1}{y} = \frac{1}{y} \left( \frac{\Delta y}{y} \right) \)

The **proportional** uncertainty in the inverse of a number is the same as the **proportional** uncertainty in the number.
Division with Multiple Uncertainties
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What if both numbers have uncertainties?
Division with Multiple Uncertainties - Example

Division operates just like multiplication. By the rules for multiplication, 
\[
\Delta \left( \frac{xy}{y} \right) \approx \left( \frac{xy}{y} \right) \left( \Delta x + \Delta y \right).
\]

If we want to find the uncertainty in \( \frac{x}{y} \), we can just make a new quantity, \( w \), where \( w = \frac{1}{y} \), so that \( \frac{x}{y} = xw \), so we know that 
\[
\Delta \left( \frac{xw}{y} \right) \approx \left( \frac{xw}{y} \right) \left( \Delta x + \Delta w \right).
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By the rules for multiplication,

\[ \Delta (xy) \approx (xy) \left( \frac{\Delta x}{x} + \frac{\Delta y}{y} \right) \]
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$$w = 1/y$$

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\[ \Delta (xw) \approx (xw) \left( \frac{\Delta x}{x} + \frac{\Delta y}{y} \right) \]
\[ \Delta \left( \frac{x}{y} \right) \approx \left( \frac{x}{y} \right) \left( \frac{\Delta x}{x} + \frac{\Delta y}{y} \right) \]
By the rules for inversion, we know that,

$$\Delta w = \Delta \left(\frac{1}{y}\right) \approx \left(\frac{1}{y}\right) \frac{\Delta y}{y} = w \frac{\Delta y}{y}$$

Which could also be written as

$$\frac{\Delta w}{w} \approx \frac{\Delta y}{y}$$

So by combining these two rules we get

$$\Delta \left( xw \right) \approx \left( xw \right) \left( \frac{\Delta x}{x} + \frac{\Delta w}{w} \right)$$

$$\Delta \left( \frac{x}{y} \right) \approx \left( \frac{x}{y} \right) \left( \frac{\Delta x}{x} + \frac{\Delta y}{y} \right)$$

When dividing numbers, we add proportional uncertainties (similar to multiplication).
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When dividing numbers, we add proportional uncertainties (similar to multiplication).

Remember that if \( x \) or \( y \) can be negative, we’ll need absolute value signs around the appropriate terms, since uncertainty contributions should always be given as positive numbers.
Recap

1. When inverting a number, the proportional uncertainty stays the same.

\[ \frac{1}{0.32} = 3.125 \pm \frac{0.0625}{3.125} = 3.125 \pm 0.00019 \]
1. When inverting a number, the *proportional* uncertainty stays the same.
Recap

1. When inverting a number, the proportional uncertainty stays the same.

\[
\frac{1}{32.0 \pm 0.2} = \frac{1}{32.0} \pm \left( \frac{0.2}{32.0} \right) \left( \frac{1}{32.0} \right) \\
= 0.03125 \pm (0.00625) 0.03125 \\
\approx 0.03125 \pm 0.00019
\]
Recap - continued

When dividing numbers, we add the proportional uncertainties.

\[(2 \pm 1) \div (32 \pm 0.2) = (232.0) \pm (232.0) \left(1 + 0.2 \cdot 32.0\right) = 0.0625 \pm 0.0316 \text{ (0.5 + 0.00625)}\]

Uncertainties in final results are usually expressed to one significant figure, so the above result becomes \(0.06 \pm 0.03\).
Recap - continued

2. When dividing numbers, we add the *proportional* uncertainties.
2. When dividing numbers, we add the proportional uncertainties.

\[
\frac{(2 \pm 1)}{(32.0 \pm 0.2)} = \left( \frac{2}{32.0} \right) \pm \left( \frac{2}{32.0} \right) \left( \frac{1}{2} + \frac{0.2}{32.0} \right)
\]

\[
= 0.0625 \pm 0.0625 (0.5 + 0.00625)
\]

\[
= 0.0625 \pm 0.0316
\]

3. Uncertainties in final results are usually expressed to one significant figure, so the above result becomes

\[
0.0625 \pm 0.0316 = 0.06 \pm 0.03
\]